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# Factorization of linear elliptic boundary value problems in non cylindrical domains

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## Abstract

We present a method of factorization for linear elliptic boundary value problems considered in non cylindrical domains. We associate a control problem to the boundary value problem which regularizes it. The technique of change of variables is used to study this problem. *To cite this article: J. Henry et al., C. R. Acad. Sci. Paris, Ser. I.*

## Résumé

**Factorisation de problèmes aux limites linéaires elliptiques dans un domaine non cylindrique.** Nous présentons une méthode de factorisation de problèmes aux limites elliptiques linéaires dans un domaine non cylindrique. On associe au problème elliptique un problème de contrôle qui en fournit une régularisation. La technique de changement de variables est utilisée pour étudier ce problème de contrôle. *Pour citer cet article : J. Henry et al., C. R. Acad. Sci. Paris, Ser. I.*

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## 1. Introduction

In [1] Angel and Bellman proposed a method based on spatial invariant embedding for transforming a second-order elliptic boundary value problem in a rectangle, in a system of first-order decoupled initial

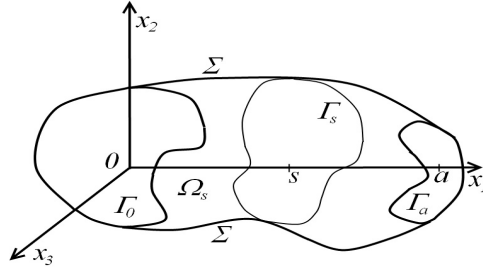
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value problems which can be solved by a two-step process. In [4], Henry and Ramos gave a complete justification for this transformation, in the case of the Poisson equation in an  $n$ -dimensional cylindrical domain. The invariant embedding was performed using the coordinate along the axis of the cylinder. The Neumann to Dirichlet (NtD) operator on a section of the cylinder was shown to satisfy a Riccati equation. The method is similar to the one used by [5], for deriving the optimal feedback for optimal control problems of parabolic equations. Its justification is based on a Galerkin method. A shorter proof was given in [2]. In this study, we generalize this method to non cylindrical domains. The relationship with a control problem is used to regularize the Riccati equation. We use a change of variables in order to set the problem in a cylindrical domain. It was shown, in the case of the cylinder, that the  $LU$  block factorization of matrix of the problem discretized by finite differences, can be viewed as a discretization of the factorized version of the boundary value problem. Other discretizations lead to new numerical schemes.

## 2. Elliptic boundary value problem in a non cylindrical domain

We denote the elements of  $\mathbb{R}^N$  by  $(x_1, x_2, \dots, x_N) = (x, y)$ , where  $x = x_1$  and  $y = (x_2, \dots, x_N)$  to stress the particular role of  $x_1$ . For each  $x \in [0, a]$ , let  $\mathcal{O}_x$  be a bounded open set in  $\mathbb{R}^{N-1}$ . We shall call a quasi-cylinder with respect to  $x$ , a set  $\Omega$  defined in  $\mathbb{R}^N$  by  $\Omega = \cup_{0 < x < a} (x, \mathcal{O}_x)$ , where  $y \in \mathbb{R}^{N-1}$  is the coordinate in the section. We make the following regularity assumption on  $\Omega$ : as in [3], we assume that each  $\mathcal{O}_x$  has a  $C^2$  boundary and that there exist  $C^2$  diffeomorphisms  $T_x$  in  $\mathbb{R}^{N-1}$ ,  $T_x(\mathcal{O}_a) = \mathcal{O}_x$ , where  $T_x$  denotes the flow associated with the speed field  $-\frac{\partial}{\partial x} T_x(y) = V(x, T_x(y))$ , continuous with respect to  $x \in [0, a]$ ,  $y \in \mathbb{R}^{N-1}$  and verifying  $T_a(y) = y$ . We also consider  $\Sigma = \cup_{0 < x < a} (x, \partial\mathcal{O}_x)$  to be the “lateral boundary” of the domain. Further,  $\Gamma_0 = \{0\} \times \mathcal{O}_0$  and  $\Gamma_a = \{a\} \times \mathcal{O}_a$  are the “faces” of the domain.



We consider the Poisson problem

$$(\mathcal{P}_0) \begin{cases} -\Delta u = -\frac{\partial^2 u}{\partial x^2} - \Delta_y u = f, & \text{in } \Omega, \\ u|_{\Sigma} = 0, \\ -\frac{\partial u}{\partial x}|_{\Gamma_0} = 0, \quad u|_{\Gamma_a} = u_1, \end{cases}$$

where  $f \in L^2(\Omega)$  and  $u_1 \in H_{00}^{1/2}(\mathcal{O}_a)$  (see [6] for the definition of this space). This problem has a unique solution in  $L^2(0, a; H^2(\mathcal{O}_x) \cap H_0^1(\mathcal{O}_x)) \cap H^1(0, a; L^2(\mathcal{O}_x))$ , which is the space of functions verifying

$$\int_0^a \|u\|_{H^2(\mathcal{O}_x)}^2 dx < \infty, \quad u|_{\partial\mathcal{O}_x} = 0 \quad \text{and} \quad \int_0^a \left( \|u\|_{L^2(\mathcal{O}_x)}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{L^2(\mathcal{O}_x)}^2 \right) dx < \infty.$$

As in [4], we want to factorize this problem by invariant embedding in the family of similar problems

$$(\mathcal{P}_{s,h}) \begin{cases} -\Delta u_s = f & \text{in } \Omega_s = \cup_{0 < x < s} (x, \mathcal{O}_x), \\ u_s|_{\Sigma_s} = 0, & -\frac{\partial u_s}{\partial x}|_{\Gamma_0} = 0, \quad u_s|_{\Gamma_s} = h, \end{cases} \quad (1)$$

where  $h \in H_{00}^{1/2}(\mathcal{O}_s)$ ,  $\Gamma_s = \{s\} \times \mathcal{O}_s$  and  $s \in ]0, a[$ .

By linearity of  $(\mathcal{P}_0)$ , for every  $s \in ]0, a[$  we define the Dirichlet to Neumann map through  $P(s)h + w = \frac{\partial u}{\partial x}|_{\Gamma_s}$ .

In Theorem 5.3 we arrive to a Riccati equation satisfied by this operator, after a change of variables.

### 3. Optimal control framework and regularization

We can formulate  $(\mathcal{P}_0)$  as an optimal control problem:

$$(\mathcal{OC}_0) \begin{cases} \frac{\partial u}{\partial x} = v & \text{in } \Omega, \quad u|_{\Sigma} = 0, \quad u|_{\Gamma_a} = u_1, \\ \inf_v \mathcal{J}(v) = \frac{1}{2} \int_{\Omega} (|\nabla_y u - \nabla_y u_d|^2 + |v|^2) dx dy, \end{cases}$$

where  $u$  stands for the state,  $v$  for the control and  $u_d$  is given almost everywhere by

$$\begin{cases} -\Delta_y u_d(x) = f(x) & \text{in } \mathcal{O}_x, \\ u_d|_{\partial \mathcal{O}_x} = 0. \end{cases}$$

This is an ill-posed problem, since  $\mathcal{J}$  is not defined for every  $v \in L^2(\Omega)$ .

In order to overcome this difficulty, we make a parabolic regularization of  $(\mathcal{OC}_0)$ :

$$(\mathcal{OC}_{\varepsilon}) \begin{cases} \frac{\partial u_{\varepsilon}}{\partial x} + \sqrt{\varepsilon} \Delta_y u_{\varepsilon} = v & \text{in } \Omega, \quad u_{\varepsilon}|_{\Sigma} = 0, \quad u_{\varepsilon}|_{\Gamma_a} = u_1, \\ \inf_v \mathcal{J}(v) = \frac{1}{2} \int_{\Omega} (|\nabla_y u_{\varepsilon} - \nabla_y u_d|^2 + |v|^2) dx dy. \end{cases}$$

We can prove (see [5]) that this control problem has a unique solution,  $u_{\varepsilon}$ , in  $L^2(0, a; H_0^1(\mathcal{O}_x))$ .

The adjoint state  $p_{\varepsilon}$  is given by:

$$\begin{cases} -\frac{\partial p_{\varepsilon}}{\partial x} + \sqrt{\varepsilon} \Delta_y p_{\varepsilon} = -\Delta_y u_{\varepsilon} - f, & \text{in } \Omega, \\ p_{\varepsilon}|_{\Sigma} = 0, \quad p_{\varepsilon}|_{\Gamma_0} = 0, \end{cases}$$

and  $v_{\varepsilon} = -p_{\varepsilon}$  is the optimality condition.

The problem  $(\mathcal{OC}_{\varepsilon})$  corresponds to the regularization of  $(\mathcal{P}_0)$  with a 4th-order operator in  $y$ :

$$(\mathcal{P}_{\varepsilon}) \begin{cases} -\frac{\partial^2 u_{\varepsilon}}{\partial x^2} - \Delta_y u_{\varepsilon} + \varepsilon \Delta_y^2 u_{\varepsilon} = f, & \text{in } \Omega, \\ u_{\varepsilon}|_{\Sigma} = 0, \quad -\frac{\partial u_{\varepsilon}}{\partial x}|_{\Gamma_0} = 0, \quad u_{\varepsilon}|_{\Gamma_a} = u_1, \\ (\frac{\partial u_{\varepsilon}}{\partial x} + \sqrt{\varepsilon} \Delta_y u_{\varepsilon})|_{\Sigma} = 0. \end{cases}$$

### 4. Change of variables

The derivation with respect to  $s$  of the invariant embedding (1), as done in [4], is not obvious as  $P$  acts on a space of functions depending on  $s$ . To avoid this problem, we are going to make a change of coordinates,

following [3]. Using the flow  $T_x$ , the quasi-cylindrical domain  $\Omega$  can be mapped isomorphically from the cylinder  $Q = ]0, a[ \times \mathcal{O}_a$ , by the change of variables  $(x, Y) \mapsto (x, y) = (x, T_x(Y))$ ,  $Y \in \mathcal{O}_a$ . Let  $DT_x$  be the jacobian of the transformation  $T_x$  and  $J_x = \det(DT_x)$ . Set  $A(x)z = -J_x^{-1} \nabla_Y \cdot (J_x (DT_x)^{-T} (DT_x)^{-1} \nabla_Y z)$  and  $B(x)z = -(DT_x)^{-T} \nabla_Y z \cdot V \circ T_x$ , where  $z(x, Y) = u(x, T_x(Y)) = u \circ T_x(Y)$ .

Being in the framework of [3], by the change of variables the regularized problem  $(\mathcal{OC}_\varepsilon)$  becomes

$$\begin{cases} \frac{\partial z_\varepsilon}{\partial x} - \sqrt{\varepsilon} A(x) z_\varepsilon + B(x) z_\varepsilon = v \circ T_x & \text{in } Q, \\ z_\varepsilon|_{\tilde{\Sigma}} = 0, \quad z_\varepsilon|_{\Gamma_a} = u_1, \\ \inf_v \mathcal{J}(v) = \frac{1}{2} \int_Q ((DT_x^{-1}(\nabla_Y z_\varepsilon - \nabla_Y z_d))^2 + |v \circ T_x|^2) J_x dx dY, \end{cases} \quad (2)$$

with  $z_d = u_d \circ T_x$  and  $\tilde{\Sigma} = ]0, a[ \times \partial \mathcal{O}_a$ .

Now the adjoint state is given by

$$\begin{cases} -\frac{\partial q_\varepsilon}{\partial x} - \sqrt{\varepsilon} A^*(x) q_\varepsilon + B^*(x) q_\varepsilon = A(z_\varepsilon - z_d), & \text{in } Q, \\ q_\varepsilon|_{\tilde{\Gamma}_0} = 0, \quad q_\varepsilon|_{\tilde{\Sigma}} = 0, \end{cases}$$

where  $\tilde{\Gamma}_0 = \{0\} \times \mathcal{O}_a$  and the optimality condition becomes  $v \circ T_x = -q_\varepsilon$ . The invariant embedding with respect to the subdomains  $Q_s = ]0, s[ \times \mathcal{O}_a$  furnishes a linear operator  $\tilde{P}_\varepsilon(s) \in \mathcal{L}(L^2(\mathcal{O}_a), L^2(\mathcal{O}_a))$ , such that the traces on  $\{s\} \times \mathcal{O}_a$  satisfy  $-q_\varepsilon = \tilde{P}_\varepsilon z_\varepsilon + r_\varepsilon$ , so that  $z_\varepsilon$  verifies  $\frac{\partial z_\varepsilon}{\partial x} - \sqrt{\varepsilon} A(x) z_\varepsilon + B(x) z_\varepsilon = \tilde{P}_\varepsilon(x) z_\varepsilon + r_\varepsilon$ .

Then we can prove, as in [3], that  $\tilde{P}_\varepsilon$  satisfies the well posed Riccati equation in the cylinder  $Q$ , in the following sense

$$\begin{aligned} & \left( \frac{d\tilde{P}_\varepsilon}{dx} \varphi, \psi \right)_{J_x} + (\sqrt{\varepsilon} A(x) \varphi, \tilde{P}_\varepsilon(x) \psi)_{J_x} + (\sqrt{\varepsilon} \tilde{P}_\varepsilon(x) \varphi, A(x) \psi)_{J_x} - (B(x) \varphi, \tilde{P}_\varepsilon(x) \psi)_{J_x} \\ & - (\tilde{P}_\varepsilon(x) \varphi, B(x) \psi)_{J_x} + (\tilde{P}_\varepsilon(x) \varphi, \tilde{P}_\varepsilon(x) \psi)_{J_x} = (A(x) \varphi, \psi)_{J_x}, \quad \forall \varphi, \psi \in H_0^1(\mathcal{O}_a), \end{aligned}$$

with  $\tilde{P}_\varepsilon(0) = 0$ .

## 5. Convergence results

We can prove the following theorem:

**Theorem 5.1** *As  $\varepsilon \rightarrow 0$ , for  $x = s$  and a fixed  $z_\varepsilon(s) = h \in H_0^1(\mathcal{O}_a)$ , we have  $z_\varepsilon \rightarrow z$ , in  $H^1(Q_s)$ , and  $\sqrt{\varepsilon} A z_\varepsilon \rightarrow 0$ , in  $L^2(Q_s)$ .*

Then, using (2), we can prove that

**Theorem 5.2** *As  $\varepsilon \rightarrow 0$ ,  $(\frac{\partial z_\varepsilon}{\partial x} + B(x) z_\varepsilon)|_{x=s} \rightarrow (\frac{\partial z}{\partial x} + B(x) z)|_{x=s}$  in  $L^2(\mathcal{O}_a)$ . This implies that, for  $h \in H_0^1(\mathcal{O}_a)$ ,  $\tilde{P}_\varepsilon h \rightarrow \tilde{P} h$ , in  $L^2(\mathcal{O}_a)$ .*

As a consequence, we obtain the desired result:

**Theorem 5.3** *The Dirichlet-Neumann operator  $\tilde{P}$  satisfies*

$$\begin{aligned} & \left( \frac{d\tilde{P}}{dx} \varphi, \psi \right)_{J_x} - (B(x) \varphi, \tilde{P}(x) \psi)_{J_x} - (\tilde{P}(x) \varphi, B(x) \psi)_{J_x} + \\ & + (\tilde{P}(x) \varphi, \tilde{P}(x) \psi)_{J_x} = (A(x) \varphi, \psi)_{J_x}, \quad \forall \varphi, \psi \in H_0^1(\mathcal{O}_a). \end{aligned} \quad (3)$$

Problem  $(\mathcal{P}_0)$  is now equivalent to the factorized formulation, with  $\tilde{P}$  given by (3),

$$\begin{aligned} \frac{dr}{dx} - B^*r + \tilde{P}r &= -Az_d, & r(0) &= 0, \\ \frac{dz}{dx} + B(x)z - \tilde{P}z &= r, & z(a) &= u_1, \text{ in } Q. \end{aligned}$$

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## References

- [1] E. Angel, R. Bellman, Dynamic programming and partial differential equations, Academic Press, 1971.
- [2] N. Bouarroudj, J. Henry, B. Louro, M. Orey, On a direct study of an operator Riccati equation appearing in boundary value problems factorization, Applied Mathematical Sciences 46, 2 (2008), 2247–2257.
- [3] G. Da Prato, J.P. Zolésio, An optimal control problem for a parabolic equation in non cylindrical domains, Systems & Control Letters 11 (1988), 73–77.
- [4] J. Henry, A.M. Ramos, Factorization of second order elliptic boundary value problems by dynamic programming, Nonlinear Analysis. Theory, Methods & Applications 59 (2004), 629–647.
- [5] J.L. Lions, Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles, Dunod, 1968.
- [6] J.L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, vol 1, Dunod, 1968.