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Well Balanced Designs for Data Placement*

Jean-Claude Bermond[†‡], Alain Jean-Marie[‡§], Dorian Mazauric[‡],
Joseph Yu[¶]

Project-Team Coati

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Abstract: The problem we consider in this article is motivated by data placement, in particular data replication in distributed storage and retrieval systems. We are given a set V of v servers along with b files (data, documents). Each file is replicated on exactly k servers. A placement consists in finding a family of b subsets of V (representing the files) called blocks, each of size k. Each server has some probability to fail and we want to find a placement which minimizes the variance of the number of available files. It was conjectured that there always exists an optimal placement (with variance better than that of any other placement for any value of the probability of failure). We show that the conjecture is true, if there exists a well balanced design, that is a family of blocks, each of size k, such that each j-element subset of V, $1 \leq j \leq k$, belongs to the same or almost the same number of blocks (difference at most one). The existence of well balanced designs is a difficult problem as it contains as a subproblem the existence of Steiner systems. We completely solve the case $k = 2$ and give bounds and constructions for $k = 3$ and some values of v and b.

Key-words: data placement, balanced designs, Steiner systems.

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† I3S (CNRS/UNS), France
‡ Inria Sophia Antipolis - Méditerranée, France
§ LIRMM, Univ. Montpellier 2, France
¶ Department of Mathematics, UFV, Abbotsford, B.C., Canada

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Configurations bien équilibrées pour le placement de données dans les réseaux pair-à-pair

Résumé : Nous considérons un problème motivé par le placement de données, en particulier la réplication de données dans les systèmes distribués de stockage et de récupération. Étant donné un ensemble V de v serveurs et un ensemble de b fichiers (données, documents), chaque fichier est répliqué dans exactement k serveurs. Un placement est une famille de b sous-ensembles de V (représentant les fichiers) appelés blocks, chacun étant de taille k. Chaque serveur a une certaine probabilité de tomber en panne et nous cherchons un placement qui minimise la variance du nombre de fichiers disponibles. Il a été conjecturé qu'il existe toujours un placement qui est optimal quelle que soit la probabilité de panne. Nous prouvons que la conjecture est vraie s'il existe une configuration équilibrée, c'est-à-dire une famille de blocks, chacun de taille k, telle que chaque sous-ensemble de V de taille j, $1 \leq j \leq k$, appartient au même nombre ou au quasi même nombre de blocks (différence d'au plus un). L'existence de configurations équilibrées est un problème difficile car il inclut comme sous-problème l'existence de systèmes de Steiner. Nous résolvons complètement le cas $k = 2$ et nous prouvons des bornes et des constructions pour $k = 3$ et certaines valeurs de v et de b.

Mots-clés : placement de données, configurations équilibrées, systèmes de Steiner.

Contents

1 Introduction

The problem we consider in this article is motivated by data placement in particular data replication in distributed storage and retrieval systems (see [1–3, 12]). We use here the terminology of design and graph theory (so the notations are somewhat different from the papers mentioned above). We are given a set V of v servers along with b files (data, documents). Each file is replicated (placed) on exactly k servers. The set of servers containing file i is therefore a subset of size k, which will be called a block and denoted B_i. A placement consists of giving a family $\mathcal{F}$ of blocks $B_i, 1 \leq i \leq b$.

A server is available (on-line) with some probability δ and so unavailable (offline, failed) with the probability $1-\delta$. The file i is said to be available if one of the servers containing it is available or equivalently the file is unavailable if all the servers containing it are unavailable. In [2, 3, 12] the authors studied the random variable Λ, the number of available files and they proved that the mean is $E(\Lambda) = b(1-(1-\delta)^k)$; so this mean is independent of the placement. However they proved that the variance of Λ depends on the placement and showed (see [12]) that minimizing the variance corresponds to minimizing the polynomial $P(\mathcal{F}, x) = \sum_{j=0}^{k} v_j x^j$ where $x = \frac{1}{1-\delta}$ (so $x \geq 1$) and v_j denotes the number of ordered pairs of blocks intersecting in exactly j elements. So we can summarize our problem as follows:

Problem: 1 ***Let*** v, k, b ***be given positive integers and*** x ***be a real number,*** $x \geq 1$***; find a placement that is a family*** $\mathcal{F}$ ***of*** b ***blocks, each of size*** k***, on a set of*** v ***elements, which minimizes the polynomial*** $P(\mathcal{F}, x) = \sum_{j=0}^{k} v_j x^j$***, where*** v_j ***denotes the number of ordered pairs of blocks intersecting in exactly*** j ***elements. Such a placement will be called optimal for the value*** x***.***

In [12] the following conjecture is proposed:

Conjecture 1 *For any* v, k, b *there exists a family* $\mathcal{F}^*$ *which is optimal for all the values of* $x \geq 1$ *(that is* $P(\mathcal{F}^*, x) \leq P(\mathcal{F}, x)$ *for any* $\mathcal{F}$ *and any* $x \geq 1$*).*

Note that for $x = 1$, we have $P(\mathcal{F}, 1) = b(b-1)$ as the value is the number of ordered pairs of blocks. So we can restrict to the case $x > 1$. Note also that all the coefficients are even; indeed if B and B' intersect in j elements, then so do B' and B. So, we could have considered only (non ordered) pairs of blocks, in which case the polynomial will have been one half of that for ordered pairs.

Before stating our results let us give some examples. Let $v = 4$, $b = 4$, $k = 2$. We can consider different placements:

- Family $\mathcal{F}_1$: $B_1 = B_2 = B_3 = B_4 = \{1, 2\}$; then $P(\mathcal{F}_1, x) = 12x^2$
- Family $\mathcal{F}_2$: $B_1 = B_2 = \{1, 2\}, B_3 = B_4 = \{3, 4\}$; then $P(\mathcal{F}_2, x) = 4x^2 + 8$
- Family $\mathcal{F}_3$: $B_1 = \{1, 2\}, B_2 = \{1, 3\}, B_3 = \{1, 4\}, B_4 = \{2, 3\}$; then $P(\mathcal{F}_3, x) = 10x + 2$
- Family $\mathcal{F}_4$: $B_1 = \{1, 2\}, B_2 = \{2, 3\}, B_3 = \{3, 4\}, B_4 = \{1, 4\}$; then $P(\mathcal{F}_4, x) = 8x + 4$.

For any $x \geq 1$, $P(\mathcal{F}_4, x) \leq P(\mathcal{F}_i, x)$ and it can be proven that indeed $\mathcal{F}_4$ is an optimal family for any $x \geq 1$. Note that according to the values of x, $\mathcal{F}_2$ can be better (or worse) than $\mathcal{F}_3$. For $x \leq \frac{3}{2}$, $P(\mathcal{F}_2, x) \leq P(\mathcal{F}_3, x)$ (for example for $x = \frac{5}{4}$, $P(\mathcal{F}_2, \frac{5}{4}) = 14 + \frac{1}{4}$ and $P(\mathcal{F}_3, \frac{5}{4}) = 14 + \frac{1}{2}$). But for $x \geq \frac{3}{2}$, $P(\mathcal{F}_2, x) \geq P(\mathcal{F}_3, x)$ (for example for $x = 2$, $P(\mathcal{F}_2, 2) = 24$ and $P(\mathcal{F}_3, 2) = 22$).

Let now $v = 5$, $b = 3$, $k = 3$. We claim that the family $\mathcal{F}^*$ consisting of the three blocks $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{3, 4, 5\}$ is optimal for all $x \geq 1$. We have $P(\mathcal{F}^*, x) = 2x^2 + 4x$. Let $\mathcal{F}$ be any other family with a polynomial $P(\mathcal{F}, x) = a_3x^3 + a_2x^2 + a_1x + a_0$. As $v = 5$, there can never be two disjoint blocks; so $a_0 = 0$. Furthermore we always have $a_3 + a_2 + a_1 = b(b-1) = 6$. So $P(\mathcal{F}, x) - P(\mathcal{F}^*, x) = (x-1)(a_3x^2 + (a_3 + a_2 - 2)x)$. If $a_3 \geq 2$ (that is at least one block repeated), then $P(\mathcal{F}, x) - P(\mathcal{F}^*, x) > 0$ for any $x > 1$. If $a_3 = 0$, among 3 blocks necessarily two of them have a pair in common and so $a_2 \geq 2$ and $P(\mathcal{F}, x) - P(\mathcal{F}^*, x) \geq 0$ for all $x \geq 1$.

2 Our results

For a family $\mathcal{F}$ let $\lambda^{\mathcal{F}}_{x_1,\dots,x_j}$ (or shortly $\lambda_{x_1,\dots,x_j}$) denote the number of blocks of the family containing the j-element subset $\{x_1, \dots, x_j\}$. **A family $\mathcal{F}$ is j-balanced** if the $\lambda_{x_1,\dots,x_j}$ are all equal or almost equal, that is, if for any two j-element subsets $\{x_1, \dots, x_j\}$ and $\{y_1, \dots, y_j\}$, $|\lambda_{x_1,\dots,x_j} - \lambda_{y_1,\dots,y_j}| \leq 1$. Furthermore, **a family $\mathcal{F}$ is well balanced** if it is j-balanced for $1 \leq j \leq k$, where k is the size of the blocks.

We first show in Section 3 that $P(\mathcal{F}, x) = \Sigma_{j=1}^{k} \Sigma_{x_1,\dots,x_j} \lambda^2_{x_1,\dots,x_j}(x-1)^j - bx^k + b^2$. The form of the above polynomial enables us to prove in Section 3 that a well balanced family is also optimal and therefore Conjecture 1 is proven for the values of b, for which there exists a well balanced family.

The rest of the paper is devoted to the construction of well balanced families and so optimal ones. We consider first the case $k = 2$ (Section 4) where such families are easy to construct for any b. The cases $k > 2$, are much more complicated. Starting with $k = 3$, there are values of v and b for which there do not exist well balanced families (Section 5). In Section 6, we deal with the case $k = 3$ using results of design theory. Indeed the problem of constructing well balanced families contains as a subproblem the question of the existence of Steiner systems. Recall that a t-Steiner system (or (v, k, λ) t-design) is a family of blocks such that each t-element subset appears in exactly λ blocks (see [6,7]). In that case it is well-known that also, for $1 \leq j \leq t$ each j-element subset appears in exactly λ_j blocks, where $\lambda_j = \lambda \frac{\binom{v-j}{t-j}}{\binom{k-j}{t-j}}$. So a t-design is j-balanced for all j, $1 \leq j \leq t$. In particular, if $t = k-1$ and the blocks are repeated the same or almost the same number of times, then a k-Steiner System is also well balanced. As an example, a Steiner Triple System (STS) consists of a family of triples, such that each pair of elements appears in exactly one triple. In that case each element appears in $\frac{v-1}{2}$ triples and no triple is repeated. Therefore, an STS is a well balanced family. It is well-known that an STS exists if and only if $v \equiv 1$ or $3 \pmod 6$ and then $b = \frac{v(v-1)}{6}$. That gives some sporadic values for which there exist well balanced families. In Section 6, we construct many other families; for example we show that such families exist for any b for the values of $v \equiv 3 \pmod 6$ for which there exist a large number of disjoint Kirkman triple systems (see [14,15]). We also develop various tools and use them to solve many cases when $v = 6t + 4$ and to verify the conjecture for small values of v. Finally, in Section 7, we present some results for values of $k > 3$.

3 Properties of $P(\mathcal{F}, x)$ and well balanced families

Recall that $\lambda_{x_1,\dots,x_j}$ denotes the number of blocks of the family containing the j-element subset $\{x_1, \dots, x_j\}$. By convention $\lambda_\emptyset = b$. In this section, we express the polynomial $P(\mathcal{F}, x)$ in function of $\lambda_{x_1,\dots,x_j}$ and deduce the optimality of well balanced families.

Proposition 1 $P(\mathcal{F}, x) = \sum_{j=0}^{k} \sum_{x_1,\dots,x_j} \lambda_{x_1,\dots,x_j}(\lambda_{x_1,\dots,x_j} - 1)(x-1)^j$.

Proof $P(\mathcal{F},x) = \sum_{h=0}^{k} v_h x^h$. Let us write $P(\mathcal{F},x) = \sum_{j=0}^{k} \mu_j (x-1)^j$. Using $x^h = (x-1+1)^h = \sum_{j=0}^{h} \binom{h}{j}(x-1)^j$, we get $\mu_j = \sum_{h=j}^{k} \binom{h}{j} v_h$.
We claim that $\mu_j = \sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)$.
Indeed $\lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)$ counts the number of ordered pairs of blocks which contain $x_1,\ldots,x_j$. This number is the sum of the ordered pairs of blocks which intersect in exactly the j elements $x_1,\ldots,x_j$, plus those intersecting in exactly $j+1$ elements containing $x_1,\ldots,x_j$, plus more generally those intersecting in exactly in h elements containing $x_1,\ldots,x_j$, where, $j \le h \le k$. When we sum on all the possible j-element subsets to obtain $\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)$, we therefore get:

- the number of ordered pairs of blocks intersecting in exactly j elements, that is v_j
- plus the number of ordered pairs of blocks intersecting in exactly $j+1$ elements, which are counted $\binom{j+1}{j}$ times. Indeed, if the intersection of two blocks is $\{x_1,\ldots,x_{j+1}\}$ they are counted for all the j-element subsets included in $\{x_1,\ldots,x_{j+1}\}$ which are in number $\binom{j+1}{j}$. Therefore we have $\binom{j+1}{j} v_{j+1}$ such ordered pairs of blocks.
- plus more generally, for h, $j \le h \le k$ we count $\binom{h}{j} v_h$ ordered pairs of blocks intersecting in exactly h elements; indeed if the intersection of two blocks is $\{x_1,\ldots,x_h\}$ they are counted for all the j-element subsets included in $\{x_1,\ldots,x_h\}$, that is $\binom{h}{j}$ times.

  Therefore we get exactly μ_j which is the left-hand side of the equation of the claim. ∎

We will use the following equality intensively

$$\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j} = b\binom{k}{j}. \tag{1}$$

It follows from the fact that a given block B is counted once in all the $\lambda_{x_1,\ldots,x_j}$ such that $\{x_1,\ldots,x_j\} \subset B$ and we have $\binom{k}{j}$ such j-element subsets.

Theorem 1 $P(\mathcal{F},x) = \sum_{j=1}^{k} \sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j}(x-1)^j - bx^k + b^2$.

Proof Using Equation 1, we get $\sum_{j=0}^{k} \sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(x-1)^j = \sum_{j=0}^{k} b\binom{k}{j}(x-1)^j = b(x-1+1)^k = bx^k$. Replacing in the expression of $P(\mathcal{F},x)$ given in Proposition 1 and using the fact that $\lambda^2_{\emptyset} = b^2$ we obtain the theorem. ∎

Proposition 2 $\sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j}$ *is minimized when* $\mathcal{F}$ *is* j*-balanced.*

Proof As by Equation 1, $\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}$ is the constant $b\binom{k}{j}$, then $\sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j}$ is minimized when all the $\lambda_{x_1,\ldots,x_j}$ are equal to $r = b\binom{k}{j}/\binom{v}{j}$ if this value is an integer or are equal either to $\lfloor r \rfloor$ or $\lceil r \rceil$ otherwise. This is equivalent to say that $\mathcal{F}$ is j-balanced. ∎

So, we can state our main theorem

Theorem 2 *If* $\mathcal{F}^*$ *is well balanced, then* $\mathcal{F}^*$ *is optimal, that is,* $P(\mathcal{F}^*,x) \le P(\mathcal{F},x)$ *for any* $\mathcal{F}$ *and any* $x \ge 1$.

Proof If $\mathcal{F}^*$ is well balanced, then all the coefficients of the polynomial as expressed in the Theorem 1 are minimized and so $\mathcal{F}^*$ is optimal. ■

Note that for a j-balanced family, the coefficient of $(x-1)^j$ in the polynomial $P(\mathcal{F}, x)$ is easy to compute. Let $b\binom{k}{j} = q\binom{v}{j} + r$, with $r < \binom{v}{j}$. Then we have r values of the $\lambda_{x_1,\dots,x_j}$ equal to $q+1$ and $\binom{v}{j} - r$ equal to q. So, $\sum_{x_1,\dots,x_j} \lambda^2_{x_1,\dots,x_j} = \binom{v}{j}q^2 + 2qr + r$.

When $b = \binom{v}{k}$, the family consisting of all the possible k-element subsets is well balanced and will be called **a complete family**. Furthermore, for any j, the values of the $\lambda_{x_1,\dots,x_j}$ are all equal to $\lambda_j = \binom{v-j}{k-j}$. By taking h copies we get also a well balanced family for $b = h\binom{v}{k}$.

Proposition 3 *Let v and k be given and let $b' = h\binom{v}{k} + b$ with $b < \binom{v}{k}$. Then, there exists a well balanced family $\mathcal{F}'$ for b' if and only if there exists a well balanced family $\mathcal{F}$ for b.*

Proof If we have a well balanced family $\mathcal{F}$ for some $b \leq \binom{v}{k}$ we can construct a well balanced family $\mathcal{F}'$ for $b' = h\binom{v}{k} + b$ by adding h complete families to $\mathcal{F}$. Conversely if we have a well balanced family $\mathcal{F}'$ for $b' = h\binom{v}{k} + b$, each k-element subset is repeated h or $h+1$ times and so by deleting h copies of each block, we can deduce a well balanced family for b. ■

The next proposition generalizes this idea to optimal families.

Proposition 4 *Let v and k be given and let $b' = h\binom{v}{k} + b$ with $b \leq \binom{v}{k}$. If there exists an optimal family for b', then there exists an optimal family for b and furthermore the optimal family for b' consists of the optimal family for b plus h complete families.*

Proof Suppose there exists an optimal family $\mathcal{F}'$ for b'. This family is necessarily k-balanced. Indeed suppose it is not the case and let $\mathcal{F}''$ be a k-balanced family (such a family can be easily constructed by taking among the $\binom{v}{k}$ subsets of size k, b of them repeated $h+1$ times and the other $\binom{v}{k} - b$ repeated h times). But, the coefficient of x^k in $P(\mathcal{F}'', x)$ will be strictly less than that of $P(\mathcal{F}', x)$ and so for x large enough $P(\mathcal{F}'', x) < P(\mathcal{F}', x)$ contradicting the optimality of $\mathcal{F}'$. So each k-element subset appears exactly h or $h+1$ times.

Now, deleting h copies of each block we get a family $\mathcal{F}$ with $b = b' - h\binom{v}{k}$ blocks (none of them being repeated). Note that if $\lambda_{x_1,\dots,x_j}$ (resp. $\lambda'_{x_1,\dots,x_j}$) denotes the number of blocks of the family $\mathcal{F}$ (resp. $\mathcal{F}'$) containing $\{x_1, \dots, x_j\}$ we have: $\lambda'_{x_1,\dots,x_j} = \lambda_{x_1,\dots,x_j} + h\binom{v-j}{k-j}$. Consider another family $\mathcal{G}$ on b blocks and let $\mathcal{G}'$ be the family on b' blocks obtained by adding h complete families to $\mathcal{G}$. Let $\mu_{x_1,\dots,x_j}$ (resp. $\mu'_{x_1,\dots,x_j}$) denote the number of blocks of the family $\mathcal{G}$ (resp. $\mathcal{G}'$) containing $\{x_1, \dots, x_j\}$. Then we have: $\mu'_{x_1,\dots,x_j} = \mu_{x_1,\dots,x_j} + h\binom{v-j}{k-j}$. So, by Equation 1, $\sum_{x_1,\dots,x_j} \lambda_{x_1,\dots,x_j} = \sum_{x_1,\dots,x_j} \mu_{x_1,\dots,x_j}$ and $\sum_{x_1,\dots,x_j} \lambda'_{x_1,\dots,x_j} = \sum_{x_1,\dots,x_j} \mu'_{x_1,\dots,x_j}$, then $\sum_{x_1,\dots,x_j} \lambda^2_{x_1,\dots,x_j} - \sum_{x_1,\dots,x_j} \mu^2_{x_1,\dots,x_j} = \sum_{x_1,\dots,x_j} \lambda'^2_{x_1,\dots,x_j} - \sum_{x_1,\dots,x_j} \mu'^2_{x_1,\dots,x_j}$ and thus $P(\mathcal{G}', x) - P(\mathcal{F}', x) = P(\mathcal{G}, x) - P(\mathcal{F}, x)$. Therefore if $\mathcal{F}$ is not optimal there exists a family $\mathcal{G}$ and a value x for which $P(\mathcal{G}, x) < P(\mathcal{F}, x)$ and for this value of x we have $P(\mathcal{G}', x) < P(\mathcal{F}', x)$ and $\mathcal{F}'$ is not optimal, a contradiction. ■

We conjecture that the converse is true: that is starting from an optimal family $\mathcal{F}$ for some $b \leq \binom{v}{k}$, the family $\mathcal{F}'$ obtained by adding h complete families is also optimal. This is true, if Conjecture 1 on the existence of an optimal family for any v, b, k is true, as in that case any optimal family is k-balanced.

In what follows we will restrict ourselves to the case $b \leq \binom{v}{k}$. In fact the following proposition shows that we only need to consider the values of $b \leq \frac{1}{2}\binom{v}{k}$.

Proposition 5 *Let v and k be given. An optimal family $\bar{\mathcal{F}}$ for $\bar{b} = \binom{v}{k} - b$ can be obtained from an optimal family $\mathcal{F}$ for $b \leq \binom{v}{k}$ by taking as blocks the k-element subsets which are not blocks of $\mathcal{F}$.*

Proof Let $\mathcal{F}$ be an optimal family with b blocks and let $\bar{\mathcal{F}}$ be the family obtained from $\mathcal{F}$ by taking as blocks the k-element subsets which are not blocks of $\mathcal{F}$. $\bar{\mathcal{F}}$ has $\bar{b} = \binom{v}{k} - b$ blocks. Furthermore, if $\bar{\lambda}_{x_1,\ldots,x_j}$ denotes the number of blocks of the family $\bar{\mathcal{F}}$ containing $\{x_1, \ldots, x_j\}$, we have $\bar{\lambda}_{x_1,\ldots,x_j} = \binom{v-j}{k-j} - \lambda_{x_1,\ldots,x_j}$. Consider another family $\bar{\mathcal{G}}$ with $\bar{b}$ blocks and let $\mathcal{G}$ be the complementary family obtained from $\bar{\mathcal{G}}$ by taking as blocks the k-element subsets which are not blocks of $\bar{\mathcal{G}}$; $\mathcal{G}$ has b blocks. We also have: $\bar{\mu}_{x_1,\ldots,x_j} = \binom{v-j}{k-j} - \mu_{x_1,\ldots,x_j}$ and so we get $P(\bar{\mathcal{G}}, x) - P(\bar{\mathcal{F}}, x) = P(\mathcal{G}, x) - P(\mathcal{F}, x)$. Therefore if $\mathcal{F}$ is an optimal family, then $\bar{\mathcal{F}}$ is also an optimal family. ∎

4 Case k = 2

Theorem 3 *Let $k = 2$. Then for any v and b there exists a well balanced family.*

Proof In view of Proposition 4, we only need to consider the case $b \leq \binom{v}{2}$. In the case $k = 2$ the blocks are pairs of elements and so the problem consists of designing a simple graph with v vertices and b edges that is almost regular (the degree of a vertex x being $d(x) = \lfloor \frac{2b}{v} \rfloor$ or $\lceil \frac{2b}{v} \rceil$). We distinguish two cases.

- Case v even. Let $b = q\frac{v}{2} + r$ for $0 \leq r < \frac{v}{2}$. It is well-known that, for v even, the edges of the complete graph K_v can be partitioned into $v - 1$ perfect matchings (set of $\frac{v}{2}$ disjoint edges covering the vertices). In that case the family consisting of q perfect matchings plus r edges of the $(q+1)$th perfect matching forms the required family with $b = q\frac{v}{2} + r$ edges, none of them repeated and with the degree of a vertex equal to q or $q + 1$.

- Case v odd. Let $b = qv + r$ for $0 \leq r < v$. It is also well-known that for v odd, the edges of the complete graph K_v can be partitioned into $\frac{v-1}{2}$ hamiltonian cycles (cycles containing each vertex exactly once). In that case consider the family consisting of q hamiltonian cycles plus the following r edges of the $(q+1)$th hamiltonian cycle: if the cycle is $x_0, x_1, \ldots, x_i, \ldots, x_{v-1}$ we take the r edges $\{x_{2j}, x_{2j+1}\}$ for $0 \leq j \leq r - 1$ (indices being taken modulo v). Then it consists of $b = qv + r$ edges none of them being repeated; furthermore the degree of a vertex is $2q$ or $2q + 1$ if $r \leq \frac{v-1}{2}$ and $2q + 1$ or $2q + 2$ otherwise and so in both cases $d(x) = \lfloor \frac{2b}{v} \rfloor$ or $\lceil \frac{2b}{v} \rceil$. ∎

An algorithm to construct a well balanced family starting from any family.

In some cases related to practical applications, files and servers may be appearing or disappearing over time, leaving the storage system in an unbalanced situation. Instead of starting over, it might be helpful to design an algorithm which, starting from some family, constructs an optimal well balanced family. That is in general a difficult problem; but for $k = 2$, we can easily design such a procedure.

Let v and b be given and $k = 2$ and consider any family $\mathcal{F}$; we will transform it into a well balanced family with the same parameters. First let us construct a 2-balanced family. Suppose, $\mathcal{F}$ is not 2-balanced; so there exist two edges (blocks) $\{x, y\}$ and $\{z, t\}$ with $\lambda_{x,y} \geq \lambda_{z,t} + 2$. Then, delete from $\mathcal{F}$ one edge $\{x, y\}$ and add one edge $\{z, t\}$. Repeating this procedure we end after a finite number of steps with a family such that for any pair of edges $\{x, y\}$ and $\{z, t\}$

$|\lambda_{x,y} - \lambda_{z,t}| \leq 1$, that is a 2-balanced family. Now let us show how to construct a well balanced family from a 2-balanced one. Let $\mathcal{F}$ be a 2-balanced family with $\lambda_{x,y} = \lambda$ or $\lambda - 1$; suppose it is not 1-balanced; then there exist two vertices x and z with $d(x) \geq d(z)+2$. So there exists a vertex $y \neq x, z$ with $\lambda_{x,y} \geq \lambda_{z,y} + 1$; otherwise $d(x) = \sum_{y \neq x,z} \lambda_{x,y} + \lambda_{x,z} \leq \sum_{y \neq x,z} \lambda_{z,y} + \lambda_{x,z} = d(z)$ a contradiction. Thus, $\lambda_{x,y} = \lambda$ and $\lambda_{z,y} = \lambda - 1$. Deleting from $\mathcal{F}$ one edge $\{x, y\}$ and adding one edge $\{z, y\}$, we still get a 2-balanced family $\mathcal{F}'$ ($\lambda'_{x,y} = \lambda - 1$ and $\lambda'_{z,y} = \lambda$); but we have reduced the gap between the degrees of x and z, as $d'(x) = d(x) - 1$ and $d'(z) = d(z) + 1$, while the other degrees remain unchanged. Repeating this procedure we end after a finite number of steps with a 1-balanced and 2-balanced, so a well balanced family.

5 Case k = 3: Impossible configurations

For $k = 3$, there are values of v and b for which there do not exist well balanced families. In this section, we identify several such sets of parameters. Then, in Section 6, we proceed towards the construction of well balanced families for some other cases.

Consider for instance $v = 4$ and $b = 2$. There are 6 possible different pairs $\{x, y\}$ and 6 pairs in the two blocks, so if there exists a 2-balanced family, then $\lambda_{x,y} = 1$ for all $\{x, y\}$. But this is impossible as $v - 1 = 3$ and there cannot exist a partition of the edges of K_4 into triples (non existence of a $(4, 3, 1)$-design). The argument is generalized in the following proposition:

Proposition 6 *Let $k = 3$, v be even and λ be odd. If $\lambda\frac{v(v-1)}{2} - \frac{v}{2} < 3b < \lambda\frac{v(v-1)}{2} + \frac{v}{2}$, then there does not exist a 2-balanced family.*

Proof Note that the number of possible pairs is $\frac{v(v-1)}{2}$. By Equation 1, $\sum_{x,y} \lambda_{x,y} = 3b$. We distinguish three cases:

- $3b = \lambda\frac{v(v-1)}{2}$. In that case a 2-balanced family will verify $\lambda_{x,y} = \lambda$ for all pairs $\{x, y\}$ and then we should have $\lambda_x = \lambda\frac{v-1}{2}$ which is impossible as λ is odd and v is even (non existence of a $(v, 3, \lambda)$-design for v even and λ odd).

- $3b < \lambda\frac{v(v-1)}{2}$. In that case we cannot have all the $\lambda_{x,y} \geq \lambda$. So we have one of the $\lambda_{x,y} \leq \lambda - 1$ and if the family is 2-balanced all the $\lambda_{x,y} \leq \lambda$. But, then $\lambda_x \leq \lambda\frac{v-1}{2}$ and as $\lambda(v - 1)$ is odd, $\lambda_x \leq \lambda\frac{v-1}{2} - \frac{1}{2}$. Using Equation 1, $3b = \sum_x \lambda_x \leq \lambda\frac{v(v-1)}{2} - \frac{v}{2}$. Therefore, there does not exist a 2-balanced family if $\lambda\frac{v(v-1)}{2} - \frac{v}{2} < 3b < \lambda\frac{v(v-1)}{2}$.

- $3b > \lambda\frac{v(v-1)}{2}$. In that case we cannot have all the $\lambda_{x,y} \leq \lambda$. So we have one of the $\lambda_{x,y} \geq \lambda + 1$ and if the family is 2-balanced all the $\lambda_{x,y} \geq \lambda$. But, then $\lambda_x \geq \lambda\frac{v-1}{2}$ and as $\lambda(v - 1)$ is odd, $\lambda_x \geq \lambda\frac{v-1}{2} + \frac{1}{2}$. Using Equation 1, $3b = \sum_x \lambda_x \geq \lambda\frac{v(v-1)}{2} + \frac{v}{2}$. Therefore there does not exist a 2-balanced family if $\lambda\frac{v(v-1)}{2} < 3b < \lambda\frac{v(v-1)}{2} + \frac{v}{2}$. ∎

For example, there do not exist well balanced families for $k = 3$ and $\{v = 6; b \equiv 5 \pmod{10}\}$; $\{v = 8; b \equiv 9, 10, 27, 28, 29, 46, 47 \pmod{56}\}$; $\{v = 10; b \equiv 14, 15, 16 \pmod{30}\}$; $\{v = 12; b \equiv 21, 22, 23 \pmod{44}\}$; $\{v = 16; b \equiv 38, 39, 40, 41, 42 \pmod{80}\}$.

Proposition 7 *Let $k = 3$. If $\lambda\frac{v(v-1)}{6}$ is not an integer, then there does not exist a well balanced family for $b = \lfloor\lambda\frac{v(v-1)}{6}\rfloor$ and $b' = \lceil\lambda\frac{v(v-1)}{6}\rceil$.*

Proof Let $b = \lfloor\lambda\frac{v(v-1)}{6}\rfloor$. If $\lambda\frac{v(v-1)}{6}$ is not an integer, then $3b = \lambda\frac{v(v-1)}{2} - \epsilon$ where $\epsilon = 1$ or 2. By Equation 1, $3b = \sum_x \lambda_x$ and so if $\mathcal{F}$ is 1-balanced $\lambda_x = \frac{\lambda(v-1)}{2}$ except for ϵ vertices for which the value is one less. Similarly by Equation 1, $3b = \sum_{x,y} \lambda_{x,y}$ and so if $\mathcal{F}$ is 2-balanced $\lambda_{x,y} = \lambda$ except for ϵ pairs appearing $\lambda - 1$ times. But for an x_0 with $\lambda_{x_0} = \frac{\lambda(v-1)}{2} - 1$, we have $\lambda(v-1) - 2$ pairs containing it (2 pairs per block containing it) and so two pairs appear $\lambda - 1$ times. If $\epsilon = 2$ we have another vertex x'_0 with $\lambda_{x'_0} = \frac{\lambda(v-1)}{2} - 1$ and altogether at least 3 pairs appear $\lambda - 1$ times (only the pair $\{x_0, x'_0\}$ can be counted twice). So, we have, in all cases, at least $\epsilon + 1$ pairs appearing $\lambda - 1$ times, contradicting the fact that if $\mathcal{F}$ is 2-balanced only ϵ pairs appear $\lambda - 1$ times.

The proof for $b' = \lceil \lambda \frac{v(v-1)}{6} \rceil$ is similar. In that case $3b' = \lambda \frac{v(v-1)}{2} + \epsilon$ where $\epsilon = 1$ or 2. If $\mathcal{F}$ is 1-balanced $\lambda_x = \frac{\lambda(v-1)}{2}$ except for ϵ vertices for which the value is one more. If $\mathcal{F}$ is 2-balanced $\lambda_{x,y} = \lambda$ except for ϵ pairs appearing $\lambda + 1$ times. The argument applied for the vertex x_0 (or both x_0 and x'_0), with $\lambda_{x_0} = \frac{\lambda(v-1)}{2} + 1$ gives at least $\epsilon + 1$ pairs appearing $\lambda + 1$ times, a contradiction. ■

Proposition 7 applies for $v \equiv 5 \pmod 6$ and $\lambda \not\equiv 0 \pmod 3$; for example there do not exist well balanced families for $\{v = 5; b \equiv 3, 4, 6, 7 \pmod{10}\}$ or $\{v = 11; b \equiv 18, 19, 36, 37 \pmod{55}\}$. It applies also for $v \equiv 2 \pmod 6$ and $\lambda \not\equiv 0 \pmod 3$; for λ odd it is included in Proposition 6, but for λ even we get new values of non existence of well balanced families for $\{v = 8; b \equiv 18, 19, 37, 38 \pmod{56}\}$.

6 Case k = 3: Construction of well balanced families

6.1 Summary of the results

Our current results are summarized in Table 1. In addition to well balanced families, we provide solutions to the optimization problem stated in introduction for the "small" values of v and any value of b. To construct some well balanced families we will use some results obtained in design theory in particular on Steiner Triple Systems (see the handbook [6] for details).

The results obtained lead us to conjecture that the values excluded by Propositions 6 and 7 are the only ones for which there do not exist well balanced families.

Conjecture 2 *Let $k = 3$, there exists a well balanced family for the values of v and b different from that excluded by Propositions 6 and 7. In particular we conjecture that, if $v \equiv 1$ or 3 (mod 6), then there exists a well balanced family for any b.*

In what follows we will construct well balanced families for $b \le \binom{v}{3}$; indeed due to Proposition 3, it gives all the values of the form $b + h\binom{v}{3}$.

6.2 STS and KTS (Steiner Triple Systems and Kirkman Triple Systems)

Recall that a $(v, 3, 1)$ Steiner Triple System (STS(v) shortly) is defined as a family of triples (blocks of size 3), such that every pair of elements belongs to exactly one block ($\lambda_{x,y} = 1$). So it is 2-balanced (and also 3-balanced); it is well-known that every vertex belongs to exactly $\frac{v-1}{2}$ blocks and therefore it is well balanced. Such a design exists if and only if $v \equiv 1$ or 3 (mod 6). In that case $b = \frac{v(v-1)}{6}$.

For example, for $v = 7$, the blocks of a $(7, 3, 1)$-design are $B_i = \{i, i + 1, i + 3\}$, $0 \le i \le 6$, the numbers being taken modulo 7. For $v = 9$, we provide below two STS(9). Those are actually disjoint *Kirkman* triple systems (see the definition below).

v	b	Result
$6t$	$ht(6t-2)$	Proposition 18
$6t+1$	$ht(6t+1)$	Proposition 8
$6t+2$	$ht(6t+2)$	Proposition 18
$6t+3$, *	any	Proposition 9
$6t+4, **$	$b_{2p}-(6t+3) \le b \le b_{2p}+6t+3$	Propositions 11 and 14
$6t+4 \ge 16$, **, $2 \le p \le 3t-1$,	$b_{2p}-(12t+7) \le b \le b_{2p}+12t+7$	Propositions 12 and 13
$6t+5$	$6t^2+9t-h+3, h \le 2t+1$	Proposition 19
$6t+5$	$6t^2+9t+h'+4, h' \le 2t+2$	Proposition 19
5	$b \not\equiv 3,4,6,7 \pmod{10}$	Proposition 20
6	$b \not\equiv 5 \pmod{10}$	Proposition 20
7	any	Proposition 20
8	$b \not\equiv 9,10,18,19,27,28,$	
	$29,37,38,46,47 \pmod{56}$	Proposition 20
9	any	Proposition 9
10	$b \not\equiv 14,15,16 \pmod{30}$	Proposition 16
11	$b \not\equiv 18,19,36,37 \pmod{55}$	Proposition 20
15	any	Proposition 9
16	$b \not\equiv 38,39,40,41,42 \pmod{80}$	Proposition 17

Table 1: Well balanced families constructed for $k=3$; $t \ge 1$; $b_{2p} := 2p(3t+2)(2t+1)$, $p \ge 1$. Condition * (resp. **) = There exist $3t+1$ disjoint STS(v) one (resp. two) of them being a KTS(v). For $v \le 11$ and $v = 16$, the conditions are all also necessary by Propositions 6 and 7.

Example 1 *Two disjoint Kirkman Triple Systems for $v = 9$:*

$$\begin{array}{cccc} \{0,\infty,\infty'\} & \{0,2,5\} & \{0,3,4\} & \{0,1,6\} \\ \{1,2,4\} & \{1,3,\infty'\} & \{1,5,\infty\} & \{2,3,\infty\} \\ \{3,5,6\} & \{4,6,\infty\} & \{2,6,\infty'\} & \{4,5,\infty'\} \end{array}$$

Example 1 (a): *a Kirkman triple System K_A for $v = 9$*

$$\begin{array}{cccc} \{1,\infty,\infty'\} & \{1,3,6\} & \{1,4,5\} & \{1,2,0\} \\ \{2,3,5\} & \{2,4,\infty'\} & \{2,6,\infty\} & \{3,4,\infty\} \\ \{4,6,0\} & \{5,0,\infty\} & \{3,0,\infty'\} & \{5,6,\infty'\} \end{array}$$

Example 1 (b): *another Kirkman Triple System K_B for $v = 9$*

Using directly Steiner Triple Systems provides some sporadic values of v and b for which there exist well balanced families. We can get more values of b by considering more than one STS(v); but we have to ensure that the family is 3-balanced (that is no block is repeated). Fortunately the answer can be obtained due to the existence of families of disjoint STS(v) (see Theorem 4 below). Two STS(v) are said to be disjoint if they have no triple in common. A set of $v-2$ disjoint STS(v) is called a *large set of disjoint STS(v)* and briefly denoted by LSTS(v). An LSTS(v) can be viewed as a partition of the complete family of $\binom{v}{3}$ triples into STS(v). In 1850, Cayley showed that there are only two disjoint STS(7) and so there is no LSTS(7). The same year Kirkman showed that there exists an LSTS(9). Such an LSTS(9) is given by taking as first STS(9) that of Example 1 (a); the 6 other STS(9) are obtained from the first one by developing modulo 7 (that is applying the automorphism fixing ∞ and ∞' and mapping i to $i+1$). For example, the second STS(9) is obtained by adding 1 to each number (∞ and ∞' are invariant and $6+1=0 \pmod 7$) and given in Example 1 (b).

Due to the efforts of many authors the following theorem completely settles the existence of LSTS(v).

Theorem 4 *([14, 15, 17] (see [13] for a simple proof)) For $v \equiv 1$ or 3 (mod 6), $v > 7$, there exists an LSTS(v).*

Proposition 8 *Let $k = 3$, and $v \equiv 1$ or 3 (mod 6), $v > 7$, then there exists a well balanced family for any b multiple of $\frac{v(v-1)}{6}$.*

Proof Let $b = h\frac{v(v-1)}{6}$; $b \leq \binom{v}{3}$ or equivalently, $h \leq v - 2$. According to Theorem 4, there exists an LSTS(v), formed of $v-2$ disjoint STS(v). Then, the family consisting of any h disjoint STS(v), extracted from the LSTS(v), is well balanced (with $\lambda_{x,y} = h$ and $\lambda_x = h\frac{v-1}{2}$). For $b \geq \binom{v}{3}$ the result follows by using Proposition 3. ■

We will see in Proposition 20 that the existence of two disjoint STS(7) suffices to construct a well balanced family for $v = 7$ and any b.

When $v = 6t + 3$, there exist STS(v) which have a stronger property. The triples of the STS(v) can themselves be partitioned into $3t + 1$ classes, called *parallel classes*, where a parallel class consists of $2t + 1$ blocks forming a partition of the v elements. Such an STS(v) is called *resolvable* or a Kirkman Triple System (briefly KTS(v)). Examples 1 (a) and (b) are KTS(9) where the 4 parallel classes correspond to the 4 columns. It is well-known that a KTS(v) exists for any $v \equiv 3$ (mod 6) [16].

In our next constructions, we will need families of disjoint STS(v) containing a KTS(v). The existence of mixed STS/KTS structures has not been specifically studied in the literature and we propose some conjectures about them (Conjectures 3, 4 and 5). However we can use results on families of disjoint KTS, which have indeed been studied for a long time. A set of $v - 2$ disjoint KTS(v) is called a *large set of disjoint KTS(v)* and briefly denoted by LKTS(v). As mentioned previously, Kirkman showed in 1850 that an LKTS(9) exists and in 1974, Denniston found an LKTS(15). For $v = 9$, the LSTS(9) described above, is in fact an LKTS(9) as the resolvabilty is conserved by automorphisms. An example of a KTS(15) denoted K_A is given in Example 2 below. Developing modulo 13, that is, applying the automorphism fixing ∞ and ∞' and mapping i to $i+1$, we get 13 disjoint KTS(15) and so an LKTS(15). Example 2 (b) shows $K_B = K_A + 1$.

Example 2 *Two disjoint Kirkman Triple Systems for $v = 15$:*

$\{0,1,9\}$	$\{0,2,7\}$	$\{0,3,11\}$	$\{0,4,6\}$	$\{0,5,8\}$	$\{0,10,12\}$	$\{1,4,5\}$
$\{2,4,12\}$	$\{3,4,8\}$	$\{1,7,12\}$	$\{1,8,11\}$	$\{1,2,3\}$	$\{3,5,9\}$	$\{2,6,11\}$
$\{5,10,11\}$	$\{5,6,12\}$	$\{6,8,10\}$	$\{2,9,10\}$	$\{6,7,9\}$	$\{4,7,11\}$	$\{3,7,10\}$
$\{7,8,\infty\}$	$\{9,11,\infty\}$	$\{2,5,\infty\}$	$\{3,12,\infty\}$	$\{4,10,\infty\}$	$\{1,6,\infty\}$	$\{8,9,12\}$
$\{3,6,\infty'\}$	$\{1,10,\infty'\}$	$\{4,9,\infty'\}$	$\{5,7,\infty'\}$	$\{11,12,\infty'\}$	$\{2,8,\infty'\}$	$\{0,\infty,\infty'\}$

Example 2 (a): *a Kirkman Triple System K_A for $v = 15$*

$\{1,2,10\}$	$\{1,3,8\}$	$\{1,4,12\}$	$\{1,5,7\}$	$\{1,6,9\}$	$\{1,11,0\}$	$\{2,5,6\}$
$\{3,5,0\}$	$\{4,5,9\}$	$\{2,8,0\}$	$\{2,9,12\}$	$\{2,3,4\}$	$\{4,6,10\}$	$\{3,7,12\}$
$\{6,11,12\}$	$\{6,7,0\}$	$\{7,9,11\}$	$\{3,10,11\}$	$\{7,8,10\}$	$\{5,8,12\}$	$\{4,8,11\}$
$\{8,9,\infty\}$	$\{10,12,\infty\}$	$\{3,6,\infty\}$	$\{4,0,\infty\}$	$\{5,11,\infty\}$	$\{2,7,\infty\}$	$\{9,10,0\}$
$\{4,7,\infty'\}$	$\{2,11,\infty'\}$	$\{5,10,\infty'\}$	$\{6,8,\infty'\}$	$\{12,0,\infty'\}$	$\{3,9,\infty'\}$	$\{1,\infty,\infty'\}$

Example 2 (b): *another Kirkman Triple System K_B for $v = 15$*

Since then, many people have done some research on their existence. The more recent paper is [18] where the reader can find other references. The results to date are summarized in the following theorem:

Theorem 5 *[18, Theorems 1.1 and 3.3]*

(a) For any integer $r \in \{1, 7, 11, 13, 17, 35, 53, 67, 91, 123\} \cup \{2^{2p+1}25^q : p, q \geq 1\}$*, there exists an LKTS(v) for* $v = 3^a 5^b r \prod_{i=1}^{s}(2.13^{n_i}+1)\prod_{j=1}^{t}(2.7^{m_j}+1)$*,* $a, n_i, m_j \geq 1$ *(*$1 \leq i \leq s, 1 \leq j \leq t$*),* $b, s, t \geq 0$ *and further* $a+s+t \geq 2$ *if* $b \geq 1$ *and* $r \neq 1$*.*

(b) There exists an LKTS$(3v)$ *for* $v = \prod_{i=1}^{s}(2q_i^{n_i}+1)\prod_{j=1}^{t}(4^{m_j}-1)$ *where* $s+t \geq 1$*,* $n_i, m_j \geq 1$*,* $q_i \equiv 7 \pmod{12}$ *and* q_i *is a prime power.*

6.3 Case $v = 6t + 3$

We first propose a simple construction which gives the answer for $v = 6t+3$, when there exist families of disjoint STS(v), at least one of them being a KTS(v).

Proposition 9 *Let* $k = 3$ *and* $v = 6t+3$*. If there exists a family of* $3t+1$ *disjoint STS(v), one of them being a KTS(v), then there exists a well balanced family for any b.*

Proof By Propositions 3 and 5, we can suppose $b \leq \frac{1}{2}\binom{v}{3} = (2t+1)(3t+1)\frac{6t+1}{2}$. Let the number of blocks be $b = q(2t+1)(3t+1) + r(2t+1) + s$ with $0 \leq q \leq 3t; 0 \leq r < 3t+1; 0 \leq s < 2t+1$. Then a well balanced family for b consists of q disjoint STS(v) taken from the family avoiding the singled-out KTS(v), plus r parallel classes of the KTS(v) and s triples of the $(r+1)$th parallel class of this KTS(v). Indeed, by assumption on the family, all the triples are disjoint and so $\lambda_{x,y,z} = 0$ or 1. In each STS(v) a pair of elements appears exactly once; so $\lambda_{x,y} = q$ or $q+1$ (exactly q if $r = 0, s = 0$). In each parallel class of the KTS(v), each vertex appears exactly once; so $\lambda_x = (3t+1)q + r$ or $(3t+1)q + r + 1$ (exactly $(3t+1)q + r$ if $s = 0$). ■

Proposition 9 can be applied when there exists an LKTS(v). There is no need to have a structure as strong as this, but only $3t+1$ disjoint STS, with one of them being a KTS. We conjecture that such a structure always exists for $v = 6t+3$; this conjecture will imply Conjecture 2 for $v \equiv 3 \pmod 6$.

Conjecture 3 *For* $v = 6t+3$*, there exist* $3t+1$ *disjoint STS(v) one of them being a KTS(v).*

The following stronger conjecture is also interesting.

Conjecture 4 *For* $v = 6t+3$*, there exist an LSTS(v) such that one of its STS(v) is a KTS(v).*

6.4 Constructions for $v = 6t + 4$

In this section we present various construction techniques for the case $v = 6t+4$. We will need the existence of two disjoint KTS$(6t+3)$ denoted K_A and K_B. We illustrate them for $v = 10$ (Proposition 16) and $v = 16$ (Proposition 17) verifying Conjecture 2 for these values. For $v = 10$ we will use the two disjoint KTS(9) of Example 1 (a) and Example 1 (b). To ease the reading, we substitute ∞ with 7 and ∞' with 8.

K_A :	$\{0,7,8\}$	$\{0,2,5\}$	$\{0,3,4\}$	$\{0,1,6\}$	K_B :	$\{1,7,8\}$	$\{1,3,6\}$	$\{1,4,5\}$	$\{0,1,2\}$
	$\{1,2,4\}$	$\{1,3,8\}$	$\{1,5,7\}$	$\{2,3,7\}$		$\{2,3,5\}$	$\{2,4,8\}$	$\{2,6,7\}$	$\{3,4,7\}$
	$\{3,5,6\}$	$\{4,6,7\}$	$\{2,6,8\}$	$\{4,5,8\}$		$\{0,4,6\}$	$\{0,5,7\}$	$\{0,3,8\}$	$\{5,6,8\}$

6.4.1 Splitting Process: Construction A

The following central construction applies to a family containing a KTS($6t+3$) and adds $2(2t+1)$ blocks to it. It consists in "splitting" triples using an extra element.

Construction A. Consider a parallel class of a KTS($6t+3$) and a new element α ($= 6t+4$) and replace each of the $2t+1$ triples $\{x_j, y_j, u_j\}$ of this class ($1 \leq j \leq 2t+1$) by the 3 triples $\{x_j, y_j, \alpha\}$, $\{x_j, u_j, \alpha\}$, and $\{y_j, u_j, \alpha\}$.

For example take the KTS(9) K_A. We replace the first class consisting of the 3 blocks $\{0, 7, 8\}$, $\{1, 2, 4\}$, $\{3, 5, 6\}$ by the 9 blocks $\{0, 7, \alpha\}$, $\{0, 8, \alpha\}$, $\{7, 8, \alpha\}$, $\{1, 2, \alpha\}$, $\{1, 4, \alpha\}$, $\{2, 4, \alpha\}$, $\{3, 5, \alpha\}$, $\{3, 6, \alpha\}$, $\{5, 6, \alpha\}$.

Proposition 10 *Let $k = 3$ and $v = 6t+4$. If there exist, for $p \leq 3t+1$, $min(2p, 6t)$ disjoint STS($6t+3$) one of them being a KTS($6t+3$), then there exists a well balanced family for $b_{2p} = 2p(3t+2)(2t+1)$.*

Proof We apply Construction A for p classes of the KTS K_A by adding a new element α. As the classes are taken in the same KTS, α appears in $3p(2t+1)$ disjoint triples; so $\lambda_\alpha = 3p(2t+1)$. Furthermore each pair $\{\alpha, x\}$ appears exactly $2p$ times; so $\lambda_{\alpha,x} = 2p$. Then, for $1 \leq p \leq 3t$, we add to this modified K_A, $(2p-1)$ STS($6t+3$), that exist by hypothesis. Any $x \neq \alpha$ appears $3t+1$ times in each of these STS and $3t+1+p$ times in the modified K_A; so, $\lambda_x = 2p(3t+1)+p = 3p(2t+1)$. Each pair $\{x, y\}$ ($x \neq \alpha, y \neq \alpha$) appears exactly once in the modified K_A and in each of the other $2p-1$ STS; so, $\lambda_{x,y} = 2p$. Therefore the family constructed is well balanced. For $p = 3t+1$, the result was already known, as the family obtained is a complete family with $b_{6t+2} = (6t+2)(3t+2)(2t+1) = \binom{6t+4}{3}$. So only $2p = 6t$ disjoint STS($6t+3$) are needed. ■

We can extend Proposition 10 to get well balanced families for more values of b either by deleting or adding blocks. However, we need the existence of a second disjoint KTS($6t+3$) K_B and of other disjoint STS. We conjecture that such a structure always exists.

Conjecture 5 *For $v = 6t+3$, there exists an LSTS(v) such that two of its STS(v) are KTS(v).*

We will present now a deletion process (Deletion A-B), an addition process (Construction B), and then a construction (Construction C) useful for small values of b, not covered by the previous constructions.

6.4.2 Deletion Process: Deletion A-B

We start with the well balanced family obtained in Proposition 10 for b_{2p} with a KTS($6t+3$) K_A. We also suppose that there exists a second KTS($6t+3$), denoted K_B, and we choose it as one of the other $2p-1$ STS($6t+3$) used in the proof of the proposition. We now present a deletion process.

Deletion A-B. This construction consists in deleting a block $\{x, y, \alpha\}$ appearing in a class of K_A modified by Construction A, and some blocks of the class of K_B which contains the pair $\{x, y\}$, except precisely the block $\{x, y, z\}$ of this class. Doing so some pairs appear one less and if we delete all the blocks of the class of K_B, different from $\{x, y, z\}$, all elements appear one less except z. We can also delete a whole class of K_B, in which case all elements appear one less except α.

We illustrate this process with an example, which will also serve for other constructions.

Example 3 (v = 10) *Consider the case $v = 10$. We will use the Deletion A-B procedure to obtain well balanced families for $b_{2p} - 12 = 30p - 12 \leq b \leq 30p = b_{2p}$ for any p, and*

$b = 30p - 13$ for $p \geq 2$. Using Proposition 5 and the fact that $\binom{10}{3} = 120$, we will also get the values $30p \leq b \leq 30p + 12$ and for $p' \leq 2$, $b' = 30p' + 13$.

We will use the two disjoint KTS(9) K_A and K_B given at the beginning of this subsection. We start with the solution obtained for $b = 30p$ in the proof of Proposition 10, where the first class of K_A is one of the modified classes in Construction A. We can apply Deletion A-B, by deleting $\{x_1, y_1, \alpha\} = \{7, 8, \alpha\}$ and some blocks of the first class of K_B (which contains the pair $\{7, 8\}$) such as $\{2, 3, 5\}$ and $\{0, 4, 6\}$, but not $\{x_1, y_1, z_1\} = \{1, 7, 8\}$. Therefore, we get a solution for $30p - 3 \leq b \leq 30p$. Note that for $b = 30p - 3$, we have $\lambda_x = 9p - 1$, except for $z_1 = 1$ ($\lambda_1 = 9p$). We can repeat the Deletion A-B by deleting $\{x_2, y_2, \alpha\} = \{1, 4, \alpha\}$ and some blocks of the third class of K_B except $\{x_2, y_2, z_2\} = \{1, 4, 5\}$. Then we repeat Deletin A-B a third time by deleting $\{x_3, y_3, \alpha\} = \{5, 6, \alpha\}$ and some blocks of the 4th class of K_B except $\{x_3, y_3, z_3\} = \{5, 6, 8\}$. So we get a well balanced family for $30p - 9 \leq b \leq 30p$; but, if $p = 1$ we cannot go further. However, if $p \geq 2$, having done 3 times the Deletion A-B, we can delete the block $\{z_1, z_2, z_3\} = \{1, 5, 8\}$ which appears in $K_C = K_A - 1$ as translated from the block $\{2, 6, 8\}$ (recall that 8 is invariant). This K_C has to be chosen as one of the $2p$ STS$(6t + 3)$ in the proof of Proposition 10 and therefore, we get a solution, when $p \geq 2$ for $b = 30p - 10$, where $\lambda_x = 9p - 3$ and $\lambda_{x,y} = 2p$ or $2p - 1$. Deleting the blocks of the 2nd class, we finally get a well balanced family for $30p - 13 \leq b \leq 30p$ for $p \geq 2$.

However, we could have done the deletion in a different way; indeed we can delete the 3 blocks appearing in the first modified class in Construction A: $\{0, 8, \alpha\}$, $\{1, 4, \alpha\}$, $\{3, 5, \alpha\}$. To avoid deleting twice some pairs we should keep the blocks of K_B: $\{0, 3, 8\}$ and $\{1, 4, 5\}$, both of which appear in the third class of K_B and $\{2, 3, 5\}$ appearing in the first class. Now, if we delete the block $\{2, 6, 7\}$ in the third class and all the blocks of the 2nd and 4th class we get a solution for $b = 30p - 10$ and deleting the two blocks of the first class different from $\{2, 3, 5\}$ we get finally a solution for $30p - 12 \leq b \leq 30p$ for any p. Using Proposition 5 and the fact that $\binom{10}{3} = 120$, we get also the values $30p' \leq b \leq 30p' + 12$ and for $p' \leq 2$, $b = 30p' + 13$.

In summary, we get, for $v = 10$, all the values of b, except $b \equiv 14, 15, 16$ (mod 30), which we already know, by Proposition 6, no well balanced family can exist and $b = 17$ (and 103), for which we will prove the existence of a well balanced family later (Construction C).

In general we can do Deletion A-B h times, $h \leq 3t + 1$ (number of classes of K_B), if the two following conditions hold: (i) all the pairs $\{x, y\}$ of the deleted blocks $\{x, y, \alpha\}$ are disjoint (in order to avoid to delete twice a pair $\{x, \alpha\}$) and (ii) for any two deleted blocks $\{x, y, \alpha\}$ and $\{x', y', \alpha\}$ in K_A the two corresponding blocks, that we keep in K_B $\{x, y, z\}$ and $\{x', y', z'\}$ should satisfy $z \neq z'$.

We can fulfill these conditions for $h \leq 3$. Indeed, let the original blocks in the class we have modified in K_A be $\{x_1, y_1, u_1\}$, $\{x_2, y_2, u_2\}$ and $\{x_3, y_3, u_3\}$. For each $i = 1, 2, 3$, we can delete one of the 3 modified blocks $\{x_i, y_i, \alpha\}$ or $\{x_i, u_i, \alpha\}$ or $\{y_i, u_i, \alpha\}$. Let in K_B the blocks containing the pair $\{x_i, y_i\}$ (resp. $\{x_i, u_i\}$ or $\{y_i, u_i\}$) be $\{x_i, y_i, a_i\}$ (resp. $\{x_i, u_i, b_i\}$ or $\{y_i, u_i, c_i\}$). Note that as each pair appears exactly once in K_B, for a given i: a_i, b_i, c_i are distinct. We can choose for z_1 for example a_1; for z_2 one among a_2, b_2, c_2 distinct from z_1 (there are at least two choices) and for z_3 one among a_3, b_3, c_3 distinct from z_1 and z_2 (there is at least one choice). Doing so we get a well balanced family for $b_{2p} - 3(2t + 1) \leq b \leq b_{2p}$.

Proposition 11 *Let $k = 3$, $v = 6t + 4$, $1 \leq p \leq 3t + 1$ and $b_{2p} = 2p(3t + 2)(2t + 1)$. If there exist $min(2p + 1, 6t + 1)$ disjoint STS$(6t + 3)$, two of them being a KTS$(6t + 3)$, then there exists a well balanced family for $b_{2p} - (6t + 3) \leq b \leq b_{2p}$.*

Now to get the value $b = b_{2p} - (6t + 4)$, we have to delete the block $\{z_1, z_2, z_3\}$. If we are lucky like in the cases $v = 10$ (see Example 3) and $v = 16$ (in Appendix 1), this block is in K_B.

Otherwise we can conclude only if $p \geq 2$. Indeed, either the block $\{z_1, z_2, z_3\}$ is in a $K_C \neq K_A$ and we choose K_C as one of the $2p$ STS($6t+3$) used in the proof of Proposition 10. Otherwise, if $\{z_1, z_2, z_3\}$ is in K_A, instead of using $\{x_3, y_3, u_3\}$ we use another block $\{x_4, y_4, u_4\}$ in the class modified of K_A and select among a_4, b_4, c_4 a value distinct from z_1 and z_2. The block $\{z_1, z_2, z_4\}$ cannot be in K_A as the pair $\{z_1, z_2\}$ appears once (already in $\{z_1, z_2, z_3\}$). For $p \geq 2$, we can apply other deletions. We have to choose if possible blocks in K_B in classes not containing one of the pairs $\{z_1, z_2\}$, $\{z_1, z_3\}$, $\{z_2, z_3\}$. It is possible to do 3 more deletions for $t > 2$ (the case $t = 1$, $v = 10$ being already handled in Example 3 and $t = 2$, $v = 16$ is solved in Appendix 1) as we have at least 10 classes. Doing so we get the values $b_{2p} - (12t+7) \leq b \leq b_{2p}$. We summarize the results obtained in the following proposition.

Proposition 12 *Let $k = 3$ and $v = 6t+4 \geq 16$, $2 \leq p \leq 3t+1$ and $b_{2p} = 2p(3t+2)(2t+1)$. If there exist min$(2p+1, 6t+1)$ disjoint STS($6t+3$), two of them being a KTS($6t+3$), then there exists a well balanced family for $b_{2p} - (12t+7) \leq b \leq b_{2p}$.*

Using Proposition 5 we get also:

Proposition 13 *Let $k = 3$, $v = 6t+4$, $1 \leq p \leq 3t+1$ and $b_{2p} = 2p(3t+2)(2t+1)$. If there exist min$(2p+1, 6t+1)$ disjoint STS($6t+3$), two of them being a KTS($6t+3$), then there exists a well balanced family for $b_{2p} \leq b \leq b_{2p} + 6t + 3$ and when $t \geq 2$ and $p \neq 3t-1$ for $b_{2p} \leq b \leq b_{2p} + (12t+7)$.*

Note that, for $p = 3t+1$, we do not need the hypothesis on the existence of disjoint STS. Indeed, we can use the complete family on $6t+4$ elements where the modified K_A consists of all the blocks containing α and K_B is a KTS($6t+3$) (which always exists).

6.4.3 Addition Process: Construction B

In what follows we suppose again that $k = 3$ and $v = 6t+4$ and that there exist $2p+1$ ($p \leq 3t$) disjoint STS($6t+3$), two of them denoted K_A and K_B being KTS($6t+3$). We start with the well balanced family obtained for b_{2p} by using K_A and by choosing the other $2p-1$ STS($6t+3$) to be different from K_B. We now present an addition process called Construction B.

Construction B. Choose a class C of K_B, replace a block $\{x, y, z\}$ by the block $\{x, y, \alpha\}$ and add some of the other $2t$ blocks of this class. This construction can be combined with Construction A as long as $\{x, y\}$ is not a pair appearing in a modified block of K_A (otherwise the block $\{x, y, \alpha\}$ will be repeated).

Doing Construction B for one class C_1 of K_B is always possible as we have the freedom to choose the classes to modify in K_A. For example, we choose a triple $\{x_1, y_1, z_1\}$ in K_B to be modified and fix the class C_A of K_A containing $\{x_1, y_1\}$ as one of the non modified class in Construction A. Applying Construction B for C_1, we get a well balanced family for $b_{2p} \leq b \leq b_{2p} + 2t + 1$.

We can do Construction B a second time with another block $\{x_2, y_2, z_2\}$ chosen in another class of K_B replacing it by the block $\{x_2, y_2, \alpha\}$. We can choose this block in such a way that the pair $\{x_2, y_2\}$ also appears in the class C_A of K_A containing $\{x_1, y_1\}$. For that, let C_A be the class of K_A containing $\{x_1, y_1\}$ and let $\{z_1, x_2, a_2\}$ be the triple of this class containing z_1. Note that this triple is different from that containing $\{x_1, y_1\}$ as no block is repeated and the block $\{x_1, y_1, z_1\}$ is already in K_B. We choose as the second class C_2 of K_B that containing the pair $\{z_1, x_2\}$. Let $\{z_1, x_2, z_2\}$ be the triple containing the pair $\{z_1, x_2\}$; we apply Construction B by replacing it with $\{z_1, x_2, \alpha\}$. So, adding $\{z_1, x_2, \alpha\}$ we get a well balanced family for $b_{2p} + 2t + 2$ and by adding some of the other blocks of C_2, we get a well balanced family for

$b_{2p} + 2t + 2 \leq b \leq b_{2p} + 4t + 2$. We can then add all the blocks of a third class of K_B getting the following proposition:

Proposition 14 *Let $k = 3$, $v = 6t + 4$, $0 \leq p \leq 3t$ and $b_{2p} = 2p(3t + 2)(2t + 1)$. If there exist $2p + 1$ disjoint STS($6t + 3$), two of them being KTS($6t + 3$), then there exists a well balanced family for $b_{2p} \leq b \leq b_{2p} + 6t + 3$.*

At this point we have to be careful as $\lambda_{z_1}, \lambda_{z_2}, \lambda_\alpha$ are one less than the other λ_x and we cannot add a block $\{z_1, z_2, \alpha\}$ as the pair $\{z_1, \alpha\}$ will appear twice more than the other pairs. Therefore we have to use a different idea. But first let us give the example of $v = 10$.

Example 3 (continued) *In the case $v = 10$, Construction B will allow us to cover values of b such that $b_{2p} = 30p \leq b \leq 30p + 13 = b_{2p} + 13$ where $p = 0, 1, 2$, or $30p \leq b \leq 30p + 9$ with $p = 3$, and by complementation the values $30p' \leq b \leq 30p' - 9$ and for $p' \geq 1$, $30p' \leq b \leq 30p' - 13$.*

We again use the two disjoint KTS(9) K_A and K_B given at the beginning of this subsection. We do Construction B with the first class C_1 of K_B, modifying the block $\{x_1, y_1, z_1\} = \{0, 4, 6\}$ of the first class to $\{0, 6, \alpha\}$ and adding this modified block and $\{1, 7, 8\}$ and $\{2, 3, 5\}$. Note that the pair $\{x_1, y_1\} = \{0, 6\}$ appears in the block $\{0, 1, 6\}$ of the 4th class of K_A (class C_A). Therefore we will not modify this class in Construction A. We have $z_1 = 4$, which appears in the block $\{4, 5, 8\}$ of the 4th class of K_A. So, we choose $x_2 = 8$. We choose C_2 as the second class of K_B, replace $\{2, 4, 8\}$ by $\{4, 8, \alpha\}$ and add the two other blocks $\{1, 3, 6\}$ and $\{0, 5, 7\}$; here $z_2 = 2$. Then, adding the blocks of a class of K_B different from C_1 and C_2, we get a well balanced family for $b_{2p} = 30p \leq b \leq 30p + 9 = b_{2p} + 9$ where $p = 0, 1, 2, 3$.

We can instead apply Construction B directly to the classes C_1 and C_2 of K_B and then to a third class C_3 of K_B by replacing the block $\{2, 6, 7\}$ with $\{2, 7, \alpha\}$ (here $z_3 = y_1 = 6$) and keeping the two blocks $\{1, 4, 5\}$ and $\{0, 3, 8\}$. Note that we are lucky, as the pair $\{2, 7\}$ is in the triple $\{2, 3, 7\}$ which belongs also to the unmodified 4th class C_A of K_A. At that point we have: $b = 30p + 9$, for $p = 0, 1, 2, 3$, and $\lambda_{z_1}, \lambda_{z_2}, \lambda_{z_3}$ are one less than the other λ_x. We can now add the block $\{z_1, z_2, z_3\} = \{2, 4, 6\}$, which appears in the STS $K_C = K_A + 4$, different from K_A (it is obtained by adding 4 to $\{0, 2, 5\}$). That works if $p < 3$, as we can choose the $2p$ STS($6t + 3$) used in the proof of Proposition 10 to be K_A and $2p - 1$ blocks different from K_C. We can then add the blocks of the 4th class of K_B. In K_A we can modify by Construction A any class except the 4th one and so we can apply the construction for $p = 0, 1, 2$.

In summary, we get all the values except $b \equiv 14, 15, 16 \pmod{30}$, which we know by Proposition 6 no well balanced family can exist and $b = 17, 18, 19, 20$ (and $b = 100, 101, 102, 103$), for which we will prove the existence of a well balanced family later (Construction C). Note that the deletion process provides also solutions for $b = 18, 19, 20$ (and $b = 100, 101, 102$).

We can use this construction for $v = 6t + 4$ and for $p < 3t$. We apply Construction B to the two blocks $\{x_1, y_1, z_1\}$ and $\{z_1, x_2, z_2\}$ in K_B, where the pairs $\{x_1, y_1\}$ and $\{z_1, x_2\}$ are in the same class C_A of K_A which will be not modified in Construction A. Then we choose as third block to be modified a block $\{z_2, x_3, z_3\}$, with $z_3 = x_1$ or y_1 and such that the block $\{z_1, z_2, z_3\}$ appears in a STS K_C different from K_A (it is necessarily different from K_B as $\{z_1, z_2\}$ appears in K_B with x_2). That is always possible as $p < 3t$; indeed it suffices to choose as the $2p$ STS($6t + 3$) used in the proof of Proposition 10 both K_A and $2p - 1$ STS($6t + 3$) different from K_C and, for the modified classes of K_A, classes different from C_A and perhaps the class C'_A containing $\{z_2, x_3\}$. Then using Construction B with the classes C_1, C_2, C_3 containing respectively $\{x_1, y_1\}$, $\{z_1, x_2\}$, and $\{z_2, x_3\}$ and adding the block $\{z_1, z_2, z_3\}$ of K_C we get a solution for $b = b_{2p} + 6t + 4$. Indeed all the λ_x have been increased by exactly 3 and some $\lambda_{x,y}$ by 1. We can then continue the process easily by adding the blocks of a 4th class of K_B, obtaining a well balanced family for $b_{2p} \leq b \leq b_{2p} + 8t + 5$.

If $t = 1$ ($v = 10$), we cannot go further as we have used the 4 classes of K_B, but we know by proposition 6 that there does not exist a family for $b_{2p} + 14$.

Now, we suppose $t \geq 2$ and continue the process by applying Construction B for other classes of K_B. We have only to make sure that the pairs $\{x, y\}$ of the modified block $\{x, y, \alpha\}$ are not appearing in a modified block of K_A and are pairwise disjoint, otherwise we will have created two pairs $\{x, \alpha\}$. So, we can do Construction B with a block $\{x_4, y_4, z_4\}$ as long as x_4 and y_4 are different from the 6 elements x_1, y_1, z_1, x_2, z_2, x_3. Note that these 6 elements can appear in a pair in at most 3 blocks of a class. So an admissible pair $\{x_4, y_4\}$ always exists. This pair can be in a class of K_A different from C_A and C'_A and the construction works only for $p \leq 3t - 2$. If $t \geq 3$, a simple counting argument shows that we can find two blocks $\{x_4, y_4, z_4\}$ and $\{z_4, y_5, z_5\}$ with x_4, y_4, z_4, y_5, different from the 6 elements x_1, y_1, z_1, x_2, z_2, x_3. Note that the pair $\{z_4, y_5\}$ can appear in a class of K_A different from C_A, C'_A and the class containing $\{x_4, y_4\}$; so in general the construction works only for $p \leq 3t - 3$. For $t = 2$, that is $v = 16$, we will see in Appendix 1 that it also true. We summarize our results in the following proposition.

Proposition 15 *Let $k = 3$, $v = 6t + 4 \geq 22$, $0 \leq p \leq 3t - 3$ and $b_{2p} = 2p(3t + 2)(2t + 1)$. If there exist $2p + 1$ disjoint STS($6t + 3$) two of them being KTS($6t + 3$), then there exists a well balanced family for $b_{2p} \leq b \leq b_{2p} + 12t + 7$.*

6.4.4 Construction C

We take the blocks of an STS(v), $v \equiv 1$ or 3 (mod 6). We choose $\frac{v+1}{2}$ pairs $\{x_i, y_i\}$ $(0 \leq i \leq \frac{v-1}{2})$ covering all the elements. So, as v is odd, each element is covered once, except one x_0 which is covered twice. Then, we add the $\frac{v+1}{2}$ blocks $\{x_i, y_i, \alpha\}$. Doing so we get a well balanced family for $v + 1$ and $b = \frac{v(v-1)}{6} + \frac{v+1}{2}$; indeed $\lambda_x = \frac{v+1}{2}$ except $\lambda_{x_0} = \frac{v+1}{2} + 1$ and $\lambda_{x,y} = 1$ except for the $\frac{v+1}{2}$ chosen pairs and $\{x_0, \alpha\}$ for which the value is 2. Then we can continue adding h disjoint blocks $(1 \leq h \leq \frac{v}{3})$ as long as they are not in the STS(v), do not contain x_0 and do not contain one of the pairs for which the value $\lambda_{x,y} = 2$. We can continue the process as long as we keep the balance.

More generally, when $v + 1 = 6t + 4$, we apply Construction C starting with some STS(v) and choosing the $\frac{v+1}{2} = 3t + 2$ covering pairs in a small number of classes (only 2 if possible) of another KTS($6t + 3$). Then we can add the h blocks of a non used class replacing the block $\{x_0, y_0, z_0\}$ containing the x_0 which is repeated twice by the block $\{\alpha, y_0, z_0\}$. We get all the values of b such that $(3t + 1)(2t + 1) + 3t + 2 \leq b \leq (3t + 1)(2t + 1) + 5t + 3 = (6t + 4)(t + 1)$. Note that we have, for $b = (6t + 4)(t + 1)$: $\lambda_x = 3(t + 1)$ and $\lambda_{x,y} = 1$ or 2. We can also mix Construction C with Construction A as long as the pairs containing α are not in a modified class of K_A. We can then continue adding a new class with a block modified and so on like we did in Construction B.

Let us now show how Construction C gives the missing values, for $v = 10$.

Example 3 (end) *In the case $v = 10$, Construction C allows to cover values of $b \in \{17, 18, 19, 20\}$, and so by Proposition 5, $b \in \{100, 101, 102, 103\}$.*

We will use Construction C by choosing as first STS(9) K_A and by picking the pairs in the KTS K_B given at the beginning of this subsection. We add the triples $\{1, 8, \alpha\}$, $\{3, 5, \alpha\}$, $\{0, 6, \alpha\}$, obtained with pairs appearing in the first class of K_B. We also add $\{2, 4, \alpha\}$, $\{0, 7, \alpha\}$ using pairs appearing in the second class of K_B. We get a well balanced family for $b = 12 + 5 = 17$. Here $\lambda_x = 5$ except $\lambda_0 = 6$, as 0 appears in two added blocks; $\lambda_{x,y} = 1$ except for the 6 pairs $\{1, 8\}$, $\{3, 5\}$, $\{0, 6\}$, $\{2, 4\}$, $\{0, 7\}$, and $\{0, \alpha\}$. Then, we can add the 2 blocks of the 3rd class $\{1, 4, 5\}$

and $\{2,6,7\}$ and the block $\{3,8,\alpha\}$. Therefore we get the missing values $17 \leq b \leq 20$. Note that for $b = 20$, $\lambda_x = 6$ and $\lambda_{x,y} = 1$ or 2, as a pair appears exactly in one block of K_B.

So, we have completely solved the case $v = 10$, as summarized in the following proposition:

Proposition 16 *For $v = 10$ conjecture 2 is verified; that is there exists a well balanced family for all b, except $b \equiv 14, 15, 16 \pmod{30}$ for which such a family cannot exist.*

We also are able to completely solve the case $v = 16$. The proof of Proposition 17 is given in Appendix 1.

Proposition 17 *For $v = 16$, Conjecture 2 is verified; that is there exists a well balanced family for all b except $b \equiv 38, 39, 40, 41, 42 \pmod{80}$ for which such a family cannot exist.*

6.5 Other tools

We can also obtain results for other congruences of v.

Proposition 18 *Let $k = 3$ and $v = 6t > 6$ (resp. $v = 6t + 2$). There exists a well balanced family for $b = ht(6t - 2)$ (resp. $b = ht(6t + 2)$).*

Proof Take, as $v + 1 \equiv 1$ or $3 \pmod 6$, the blocks of a set of h disjoint STS$(v + 1)$ and delete all the $h\frac{v}{2}$ blocks containing the element $v + 1$. ∎

We can extend this construction to other values. As an example, consider $v = 8$ and $b = 12$. We start with the solution obtained before for $b = 8$ by deleting the blocks containing ∞' in the Example 1 (a) of a KTS(9). Note that $\lambda_{x,y} = 1$ except for the 4 pairs $\{0,\infty\}$, $\{1,3\}$, $\{2,6\}$, $\{4,5\}$ which are missing. We can add now 4 blocks taken from another KTS(9), for example that of Example 1 (b), containing these pairs; namely the blocks $\{5,0,\infty\}$, $\{1,3,6\}$, $\{2,6,\infty\}$, $\{1,4,5\}$.

We can also use, instead of triple systems, **packing or covering** with triples. For example, it is known (see [8]), that when $v \equiv 5 \pmod 6$, $K_v - H$, where H is a 2-regular graph can be decomposed into triples when the number of edges is a multiple of 3. In particular, if we take a cycle $H = C_{3h+1}$, $3h + 1 \leq v$, we get a well balanced family for $b = \frac{v(v-1)-6h-2}{6}$. We get more values by taking decompositions of $\lambda K_V - H$, where H is a 2-regular graph (see [4,5], but one needs to check that there are no repeated triples). Similarly (see [9]), for $v \equiv 5 \pmod 6$, $K_v + H$, where H is a 2-regular graph can be decomposed into triples if the number of edges is a multiple of 3. In particular if we take $H = C_{3h'+2}$, $3h' + 2 \leq v$ we get a well balanced family for $b = \frac{v(v-1)+6h'+4}{6}$. For example, for $v = 11$ we get a well balanced family for $b = 15, 16, 17$ and $b = 20, 21, 22$.

Proposition 19 *Let $k = 3$ and $v = 6t + 5$. Then there exists a well balanced family for $b = \frac{v(v-1)-6h-2}{6}$ with $3h + 1 \leq v$ and $b = \frac{v(v-1)+6h'+4}{6}$ with $3h' + 2 \leq v$.*

Similarly, when $v \equiv 0, 2 \pmod 6$, K_v minus a perfect matching can be decomposed into triples and so we get a well balanced family for $b = \frac{v(v-2)}{6}$ and when $v \equiv 0 \pmod 6$, K_v plus a perfect matching can be decomposed into triples and so we get a well balanced family for $b = \frac{v^2}{6}$.

6.6 Small values of v

We can apply the preceding techniques and other tools to deal with the small values of v verifying for $v \leq 11$ in Conjecture 2. We give the proofs and technical details in Appendix 2.

Proposition 20 *Let $k = 3$, for $v \leq 11$, there exists a well balanced family for the values of v and b different from that excluded by Propositions 6 and 7. In particular for $v = 7, 9$ there exists a well balanced family for any b.*

7 Case $k > 3$

We can generalize Proposition 6 in different ways. The first one concerns the non existence of 2-balanced families.

Proposition 21 *Let $\lambda(v-1) = q(k-1) + r$ with $0 < r \leq k-2$. If $\lambda v(v-1) - rv < k(k-1)b < \lambda v(v-1) + (k-1-r)v$, then there does not exist a 2-balanced family.*

Proof Note that the number of possible pairs is $\frac{v(v-1)}{2}$ and that a block contains $\frac{k(k-1)}{2}$ pairs. We distinguish 3 cases.

- $k(k-1)b = \lambda v(v-1)$. In that case a 2-balanced family will verify $\lambda_{x,y} = \lambda$ for all pairs $\{x, y\}$ and then we should have $\lambda_x = \lambda \frac{v-1}{k-1}$ impossible as $r \neq 0$ (non existence of a (v, k, λ)-design).
- $k(k-1)b < \lambda v(v-1)$. In that case, we cannot have all the $\lambda_{x,y} \geq \lambda$. So we have one of the $\lambda_{x,y} \leq \lambda - 1$ and if the family is 2-balanced all the $\lambda_{x,y} \leq \lambda$. But, then $\lambda_x \leq \lambda \frac{v-1}{k-1}$ and according to the definition of r, $\lambda_x \leq \lambda \frac{v-1}{k-1} - \frac{r}{k-1}$. Using Equation 1, $kb = \sum_x \lambda_x \leq \lambda \frac{v(v-1)}{k-1} - \frac{rv}{k-1}$. Therefore there does not exist a 2-balanced family if $\lambda v(v-1) - rv < k(k-1)b < \lambda v(v-1)$.
- The case $\lambda v(v-1) < k(k-1)b < \lambda v(v-1) + (k-1-r)v$ can be handled exactly as the preceding one. ■

We can also generalize Proposition 6 to ensure non existence of p-balanced families $p > 2$. We give the result for $p = 3$.

Proposition 22 *Let $\lambda_3(v-2) = q(k-2) + r$ with $0 < r \leq k-3$. If $\lambda_3 v(v-1)(v-2) - rv(v-1) < k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2) + (k-2-r)v(v-1)$, then there does not exist a 3-balanced family.*

Proof Note that the number of possible triples is $\frac{v(v-1)(v-2)}{6}$ and that a block contains $\frac{k(k-1)(k-2)}{6}$ triples. We distinguish 3 cases.

- $k(k-1)(k-2)b = \lambda_3 v(v-1)(v-2)$. In that case a 3-balanced family will verify $\lambda_{x,y,z} = \lambda_3$ for all triples $\{x, y, z\}$ and then we should have $\lambda_{x,y} = \lambda_3 \frac{v-2}{k-2}$ impossible as $r \neq 0$ (non existence of a (v, k, λ_3) 3-design).
- $k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2)$. In that case we cannot have all the $\lambda_{x,y,z} \geq \lambda_3$. So we have one of the $\lambda_{x,y,z} \leq \lambda_3 - 1$ and if the family is 3-balanced all the $\lambda_{x,y,z} \leq \lambda_3$. But, then $\lambda_{x,y} \leq \lambda_3 \frac{v-2}{k-2}$ and according to the definition of r, $\lambda_{x,y} \leq \lambda_3 \frac{v-2}{k-2} - \frac{r}{k-2}$. Using Equation 1, $\frac{k(k-1)}{2} b = \sum_{xy} \lambda_{x,y} \leq \lambda_3 \frac{v(v-1)(v-2)}{2(k-2)} - \frac{rv(v-1)}{2(k-2)}$. Therefore there does not exist a 3-balanced family if $\lambda_3 v(v-1)(v-2) - rv(v-1) < k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2)$.

- The case $\lambda_3 v(v-1)(v-2) < k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2) + (k-2-r)v(v-1)$ can be handled exactly as the preceding one. ∎

We could also get a similar result by using for a 3-balanced family the values of λ_x but the result is in fact a consequence of Propositions 21 and 22.

Consider for example $k = 4$ and $v = 9$. By Proposition 21, with $\lambda = 1$ there does no exist a 2-balanced family for $b = 5, 6$ and with $\lambda = 2$ for $b = 12, 13$; and more generally for $b \equiv 5, 6, 12, 13 \pmod{18}$. By Proposition 22, with $\lambda_3 = 1$ there does no exist a 3-balanced family for $b = 19, 20, 21, 22, 23$ and with $\lambda_3 = 3$ for $b = 61, 62, 63, 64, 65$ and more generally for $b \equiv 19, 20, 21, 22, 23 \pmod{42}$.

On the constructive side we have seen in Section 3 that a $(v, k, \lambda)(k-1)$-design is a well balanced family. Recall that a (v, k, λ) t-design is a family of blocks of size k such that each t-element subset appears in exactly λ blocks. When $t = k-1$ and $\lambda = 1$ a $(v, k, 1)(k-1)$-design is also called a Steiner System $S(k-1, k, n)$. For $k = 3$ we have the classical STS(v).

For $k = 4$ it has been proved that a $(v, 4, 1)$ 3-design also called a quadruple system SQS(v) exists if and only if $v \equiv 2$ or $4 \pmod 6$ [11]. For larger values of λ see for example Table 4.37 on page 82 of [6]. For $k \geq 5$ only few Steiner systems are known (see chapter II.5 of [6]), such as the $(12, 6, 1)$5-design and the $(11, 5, 1)$4-design obtained by deleting an element or the $(24, 6, 1)$5-design and the $(23, 5, 1)$4-design.

Similar techniques as those used for $k = 3$ can be used for small values of v to obtain well balanced families for $k = 4$. We can also use resolvable designs. For $k = 4$ and $v \equiv 4$ or $8 \pmod{12}$, there exist resolvable Kirkman Quadruple Systems, that is $(v, 4, 1)(3)$-design such that the quadruples can themselves be partitioned into $\frac{(v-1)(v-2)}{6}$ parallel classes, each consisting of $\frac{v}{4}$ blocks forming a partition of the v elements. We can also use disjoint SQS(v). Two SQS(v) are said to be disjoint if they have no quadruple in common. Similarly to STS(v), a set of $v-3$ disjoint SQS(v) is called a *large set of disjoint SQS(v)* and briefly denoted by LSQS(v). Unfortunately no such system has been shown to exist. However in [10] $v-5$ disjoint quadruple systems have been exhibited when $v = 5.2^p$.

8 Conclusion

In this article we attack a conjecture (Conjecture 1) coming from a data placement problem. In this process, we introduce a new class of combinatorial objects, called well balanced families, which generalize classical designs. We give various constructions of well balanced families of triples; but we are far from getting a complete answer (see Conjecture 2). In some cases the answer will follow from some conjectures on disjoint Steiner Triple Systems which are of interest in themselves (Conjectures 3, 4, 5) and we hope that this paper will motivate new researches in design theory.

References

[1] A-E. Baert, V. Boudet, A. Jean-Marie, and X. Roche. Minimization of download times in a distributed vod system. In *ICPP08: The international conference on parallel processing*, pages 173–180, Los Alamitos, CA, USA, 2008. IEEE.

[2] A-E. Baert, V. Boudet, A. Jean-Marie, and X. Roche. Minimization of download time variance in a distributed vod system. *Scalable Computing Practice and experience*, 10(1):75–86, 2009.

[3] A-E. Baert, V. Boudet, A. Jean-Marie, and X. Roche. Performance analysis of dat replication in grid delivery networks. In *Int. Conf. on Complex Intelligent and Software Intensive Systems*, pages 369–374, 2009.

[4] J. Chaffee and C.A. Rodger. Group divisible designs with two associate classes and quadratic leaves of triple systems. *Discrete Mathematics*, 313:2104–2114, 2013.

[5] J. Chaffee and C.A. Rodger. Neighborhoods in maximum packings of $2k_n$ and quadratic leaves of triple systems. *Journal of Combinatorial Designs*, 2013. DOI: 10.1002/jcd.21374.

[6] Colbourn C.J. and Dinitz J.H., editors. *The CRC Handbook of Combinatorial Designs (2nd edition)*, volume 42. CRC Press, 2006.

[7] C. J. Colbourn and R. Mathon. Steiner systems. In C.J. Colbourn and J.H. Dinitz, editors, *The CRC Handbook of Combinatorial Designs (2nd edition)*, volume 42 of *Discrete Mathematics and Its Applications*, chapter II.5, pages 102–110. CRC Press, second edition, 2006.

[8] C.J. Colbourn and A. Rosa. Quadratic leaves of maximal partial triple systems. *Graphs Combin.*, 2:317–337, 1986.

[9] C.J. Colbourn and A. Rosa. Quadratic excesses of coverings by triple. *ArsGraphs Combin.*, 24:23–30, 1987.

[10] T. Etzion and A. Hartman. Towards a large set of Steiner quadruple systems. *SIAM J. Discrete Mathematics*, 4:182–195, 1991.

[11] H. Hanani. On quadruple systems. *Canad J. math*, pages 145–167, 1960.

[12] A. Jean-Marie, X. Roche, A-E. Baert, and V. Boudet. Combinatorial designs and availability. Technical Report RR 7119, INRIA, 2009.

[13] L. Ji. A new existence proof for large sets of disjoint Steiner triple systems. *J. Combinatorial theory A*, 112:308–327, 2005.

[14] J.X. Lu. On large sets of disjoint Steiner triple systems I, II, III. *J. Combinatorial theory A*, 34:140–146,147–155, and 156–182, 1983.

[15] J.X. Lu. On large sets of disjoint Steiner triple systems IV, V, VI. *J. Combinatorial theory A*, 37:136–163,164–188, and 189–192, 1984.

[16] D.K. Ray-Chauduri and R.M. Wilson. Solution of Kirkman's schoolgirl problem. *Proc. symp. pure Math*, 19:187–204, 1971.

[17] L. Teirlinck. A completion of Lu's determination of the spectrum of large sets of disjoint Steiner triple systems. *J. Combinatorial theory A*, 57:302–305, 1991.

[18] J. Zhou and Y. Chang. New results on large sets of Kirkman triple systems. *Des Codes Cryptogr*, 55:1–7, 2010.

Appendix 1: case $v = 16$ (proof of Proposition 17)

For $v = 16$ we will use the two disjoint KTS(15) of Example 2 (a) and Example 2 (b), denoted respectively K_A and K_B. To ease the reading, we substitute ∞ with 13 and ∞' with 14.

K_A :	$\{0,1,9\}$	$\{0,2,7\}$	$\{0,3,11\}$	$\{0,4,6\}$	$\{0,5,8\}$	$\{0,10,12\}$	$\{1,4,5\}$
	$\{2,4,12\}$	$\{3,4,8\}$	$\{1,7,12\}$	$\{1,8,11\}$	$\{1,2,3\}$	$\{3,5,9\}$	$\{2,6,11\}$
	$\{5,10,11\}$	$\{5,6,12\}$	$\{6,8,10\}$	$\{2,9,10\}$	$\{6,7,9\}$	$\{4,7,11\}$	$\{3,7,10\}$
	$\{7,8,13\}$	$\{9,11,13\}$	$\{2,5,13\}$	$\{3,12,13\}$	$\{4,10,13\}$	$\{1,6,13\}$	$\{8,9,12\}$
	$\{3,6,14\}$	$\{1,10,14\}$	$\{4,9,14\}$	$\{5,7,14\}$	$\{11,12,14\}$	$\{2,8,14\}$	$\{0,13,14\}$

K_B :	$\{1,2,10\}$	$\{1,3,8\}$	$\{1,4,12\}$	$\{1,5,7\}$	$\{1,6,9\}$	$\{0,1,11\}$	$\{2,5,6\}$
	$\{0,3,5\}$	$\{4,5,9\}$	$\{0,2,8\}$	$\{2,9,12\}$	$\{2,3,4\}$	$\{4,6,10\}$	$\{3,7,12\}$
	$\{6,11,12\}$	$\{0,6,7\}$	$\{7,9,11\}$	$\{3,10,11\}$	$\{7,8,10\}$	$\{5,8,12\}$	$\{4,8,11\}$
	$\{8,9,13\}$	$\{10,12,13\}$	$\{3,6,13\}$	$\{0,4,13\}$	$\{5,11,13\}$	$\{2,7,13\}$	$\{0,9,10\}$
	$\{4,7,14\}$	$\{2,11,14\}$	$\{5,10,14\}$	$\{6,8,14\}$	$\{0,12,14\}$	$\{3,9,14\}$	$\{1,13,14\}$

Deletion A-B.

We start with the solution obtained for $b = 80p$ in the proof of Proposition 10, where the first class of K_A is one of the p modified classes in Construction A. We apply Deletion A-B. In the first modified class of K_A, we can delete $\{0, 9, \alpha\}$, $\{4, 12, \alpha\}$, $\{7, 13, \alpha\}$. We keep in K_B, the blocks $\{0, 9, 10\}$ ($z_1 = 10$), $\{4, 12, 1\}$ ($z_2 = 1$), and $\{2, 7, 13\}$ ($z_3 = 2$) and delete the other blocks of the 7th, 3rd and 6th classes of K_B. Then we can delete the block $\{z_1, z_2, z_3\} = \{1, 2, 10\}$ which appears in the first class of K_B, getting a solution for b, such that $80p-16 \le b \le 80p$. We can continue the process with the deletion still in the first modified class of K_A of $\{5, 11, \alpha\}$ and $\{6, 14, \alpha\}$, keeping in K_B the blocks $\{5, 11, 13\}$ ($z_4 = 13$) and $\{6, 8, 14\}$ ($z_5 = 8$), and deleting the other blocks of the 5th and 4th class of K_B.

At that point, we can either delete all the blocks of the second class of K_B and so we get the values $80p - 31 \le b \le 80p$. Or we can delete the blocks in another modified class of K_A, but we need $p \ge 2$. If $p \ge 2$, we can delete the block $\{4, 5, \alpha\}$ appearing in the 7th modified class of K_A and delete the blocks of the second class of K_B except $\{4, 5, 9\}$ ($z_6 = 9$). Then we can delete the block $\{z_4, z_5, z_6\} = \{8, 9, 13\}$ of the first class of K_B, getting a solution for $80p - 32$. Finally, we can also delete the 3 remaining blocks of the first class of K_B getting a solution for $80p - 35 \le b \le 80p$ ($p \ge 2$).

Using Proposition 5 and the fact that $\binom{16}{3} = 560$, we get also the values $80p' \le b \le 80p' + 31$ and for $p' \le 5$, $80p' \le b \le 80p' + 35$. So, in summary, we get all the values except $b \equiv 38, 39, 40, 41, 42 \pmod{80}$, for which we know by Proposition 6 that no well balanced family can exist, $b \equiv 36, 37, 43, 44 \pmod{80}$ for which we will prove the existence by the addition process and by Construction C, and $b = 45, 46, 47, 48$ (and $b = 512, 513, 514, 515$) for which we will use Construction C.

Construction B. We suppose here that the 7th class of K_A is not modified in construction A. Now we apply Construction B, by replacing the block $\{8, 9, 13\}$ of the first class of K_B by $\{8, 9, \alpha\}$ ($x_1 = 8$, $y_1 = 9$, and $z_1 = 13$), then the block $\{0, 4, 13\}$ of the 4th class of K_B by $\{0, 13, \alpha\}$ ($x_2 = 0$, $z_2 = 4$) and the block $\{4, 5, 9\}$ of the second class of K_B by $\{4, 5, \alpha\}$, ($x_3 = 5$, $z_3 = 9$), and adding the other blocks of the first, second and 4th class. Note that the pairs $\{8, 9\}$, $\{0, 13\}$, $\{4, 5\}$ are in the same class of K_A, namely the 7th class. So we get a well balanced family for $80p \le b \le 80p + 15$ for $p \le 6$.

For $p \le 5$, we add the block $\{z_1, z_2, z_3\} = \{4, 9, 13\}$ which appears in the KTS $K_A + 3$ as translated from the block $\{1, 6, 13\}$ (recall that 13 is invariant) and also the blocks of another class. Doing so we get all the $80p \le b \le 80p + 21$ ($p \le 5$).

At that point we use a variant of Construction B; indeed instead of choosing the pair $\{x_4, y_4\}$ in a class of K_B different from one already used, we can choose it in a modified class add the remaining blocks of this class and all the blocks of another class to keep the balance. For example we replace $\{3, 10, 11\}$ of the 4th class by $\{3, 10, \alpha\}$ ($z_4 = 11$), and $\{6, 11, 12\}$ of the first class by $\{6, 11, \alpha\}$ ($z_5 = 12$). The advantage is that the pairs $\{3, 10\}$, $\{6, 11\}$ are still in the 7th class of K_A. Adding the blocks of 2 other classes we get a well balanced family for all $80p \leq b \leq 80p+31$ (we have to choose in K_A not to modify the 7th class in Construction A). We can then replace the block $\{3, 7, 12\}$ of the 7th class of K_B by $\{7, 12, \alpha\}$ ($z_6 = 3$) and add all the other blocks of the 7th class of K_B. We should not modify in K_A the class containing $\{7, 12\}$ namely the 3rd one. That is possible; indeed, as $p \leq 5$, we can leave 2 classes unmodified in K_A (the 3rd and 7th). Then we can add the block $\{z_4, z_5, z_6\} = \{3, 11, 12\}$ which appears in $K_A + 8$ as translated from the block $\{8, 3, 4\}$ and finally blocks of the last class not used of K_B.

In summary we get a well balanced family for $80p \leq b \leq 80p + 37$ for $0 \leq p \leq 5$ and for $p = 6$, only $480 \leq b \leq 495$. Using Proposition 5 and the fact that $\binom{16}{3} = 560$ we get also the values $65 \leq b \leq 80$ and for $p' \geq 2$, $80p' - 37 \leq b \leq 80p'$. So, we get all the values except $b \equiv 38, 39, 40, 41, 42 \pmod{80}$, for which we know by Proposition 6 that no well balanced family can exist, and $43 \leq b \leq 64$ (and $496 \leq b \leq 517$) for which we will use Construction C. For $49 \leq b \leq 64$ (and $496 \leq b \leq 511$), it follows also from the process of deletion.

Construction C. We use the Construction C by choosing the STS(15) K_A and by picking the pairs in the KTS K_B. We add the triples $\{0, 5, \alpha\}$, $\{11, 12, \alpha\}$, $\{8, 9, \alpha\}$ and $\{4, 7, \alpha\}$ obtained with pairs appearing in the first class of K_B. We also add $\{1, 3, \alpha\}$, $\{0, 6, \alpha\}$, $\{10, 13, \alpha\}$ and $\{2, 14, \alpha\}$ with pairs appearing in the second class of K_B. We get a well balanced family for $b = 35 + 8 = 43$. Here $\lambda_x = 8$ except $\lambda_0 = 9$, as 0 appears in two pairs. Then, we can add the blocks of the 4th class replacing $\{0, 4, 13\}$ by $\{4, 13, \alpha\}$ and so we get the missing values $43 \leq b \leq 48$. That is enough to conclude.

Remark : However if we do not want to use DeletionA-B , but use only the addition process, we can get the missing values $49 \leq b \leq 64$ by pursuing construction C. Note that for $b = 48$, $\lambda_x = 9$ and $\lambda_{x,y} = 1$ or 2 as a pair appears exactly in one block of K_B. We then add the blocks of the 3rd (resp. 5th and 6th) classes of K_B, replacing $\{7, 9, 11\}$ by $\{7, 11, \alpha\}$, (resp. $\{2, 3, 4\}$ by $\{2, 3, \alpha\}$ and $\{5, 8, 12\}$ by $\{8, 12, \alpha\}$). Finally, we can add the block $\{4, 5, 9\}$ which appears in the second class of K_B and has not been modified. Note that $\lambda_x = 12$ and no pairs appears 3 times as each element appears with α one or twice (case of $0, 2, 3, 4, 7, 8, 11, 12, 13$). Therefore, we get all the values $43 \leq b \leq 64$. So we have completely solved the case $v = 16$.

Appendix 2: Small cases (proof of Proposition 20)

The case $v = 9$ was settled in Proposition 9 and $v = 10$ in Proposition 16.

v = 5. For $v = 5$, $\binom{5}{3} = 10$ and by Proposition 5 we have to consider only the values of $b \leq 5$.

We have well balanced families for $b = 1$ (one block) and $b = 2$ (two blocks $\{1, 2, 3\}$ and $\{1, 4, 5\}$), but not for $b = 3$ as we have seen in the example of the introduction (see also Proposition 7). However there exists an optimal solution $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{3, 4, 5\}$ 1-balanced but not 2-balanced ($\lambda_{1,2} = 2$ but $\lambda_{1,5} = \lambda_{2,5} = 0$). By Proposition 7, there is no well balanced solution for $b = 4$; an optimal one consists of the blocks $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{3, 4, 5\}$. For $b = 5$ there exists a well balanced solution with $\lambda_x = 3$ and $\lambda_{x,y} = 1$ or 2 and consisting of the 5 blocks $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{3, 4, 5\}$, $\{2, 4, 5\}$.

v = 6. For $v = 6$, $\binom{6}{3} = 20$ and by Proposition 5 we need to consider only the values of $b \leq 10$.

For $b = 5$ (and so $b = 15$), there does not exist a well balanced family (Proposition 6). An optimal solution $\mathcal{F}^*$ consists of the 5 blocks: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,5,6\}$, $\{2,5,6\}$, $\{3,4,5\}$ ($\lambda_x = 2$ or 3 and $\lambda_{1,2} = \lambda_{5,6} = 2$ but $\lambda_{3,6} = \lambda_{4,6} = 0$) with $P(\mathcal{F}^*, x) = 4x^2 + 16x$ as associated polynomial. The proof is obtained by inspection of the different possible cases. Proposition 4 in [12] also allows us to conclude directly for this case.

For the other values of b, we can construct well balanced families as follows. Let $B_1 = \{1,2,3\}$, $B_2 = \{4,5,6\}$; $C_1 = \{1,2,4\}$, $C_2 = \{1,3,5\}$, $C_3 = \{2,3,6\}$; $D_1 = \{1,4,6\}$, $D_2 = \{2,5,6\}$, $D_3 = \{3,4,5\}$ and $C'_1 = \{1,2,5\}$, $C'_2 = \{1,3,6\}$, $C'_3 = \{2,3,4\}$. Note that the C_i and C'_i (resp. D_i) intersect B_1 (resp. B_2) in three different pairs and B_2 (resp. B_1) in 3 different elements. Solutions are obtained by taking: for $b = 1$, B_1; for $b = 2$, B_1, B_2; for $b = 3$, C_1, C_2, C_3; for $b = 4$, C_1, C_2, C_3, B_2; for $b = 6$, C_1, C_2, C_3, D_1, D_2, D_3; for $b = 7$, C_1, C_2, C_3, D_1, D_2, D_3, B_1; for $b = 8$, C_1, C_2, C_3, D_1, D_2, D_3, B_1, B_2; for $b = 9$, C_1, C_2, C_3, D_1, D_2, D_3, C'_1, C'_2, C'_3; for $b = 10$, C_1, C_2, C_3, D_1, D_2, D_3, C'_1, C'_2, C'_3, B_2.

v = 7. For $v = 7$, $\binom{7}{3} = 35$; by Proposition 3 and Proposition 5 we have to consider only the values of $b \leq 17$. Kirkman proved that there exist two disjoint STS(7). The first one consists of the 7 blocks $C_i = \{i, i+1, i+3\}$, for $0 \leq i \leq 6$ and the second one of the 7 blocks $D_i = \{i, i+2, i+3\}$, for $0 \leq i \leq 6$ (indices modulo 7). Let $B_1 = \{1,2,3\}$, $B_2 = \{4,5,6\}$, $B_3 = \{0,1,4\}$, $B_4 = \{0,2,5\}$, $B_5 = \{0,3,6\}$. Note that these 5 blocks are disjoint from the blocks C_i and D_i. For $b = j$, $1 \leq j \leq 5$, take the blocks B_i, $1 \leq i \leq j$. For $b = 7$ take the first STS(7) (that is all the C_i). For $b = 6$ delete one block from the STS(7). For $b = 7 + j$, $1 \leq j \leq 5$ add to the STS(7) the blocks B_i, $1 \leq i \leq j$. For $b = 14$ take the two disjoint STS(7) (that is all the C_i and D_i). For $b = 13$ delete one block from one STS(7). For $b = 14 + j$, $1 \leq j \leq 5$ add to the two disjoint STS(7) the blocks B_i, $1 \leq i \leq j$.

v = 8. For $v = 8$, $\binom{8}{3} = 56$ and by Proposition 5 we have to consider only the values of $b \leq 28$. By Proposition 6 and 7 there do not exist well balanced families for $b = 9, 10, 18, 19, 27, 28$. For the other values let us construct a well balanced family.

Cases $1 \leq b \leq 8$. By Proposition 18, we have a solution for $b = 8$, consisting of the 8 blocks obtained by deleting 8 in the KTS(9) K_A (see Section 6.4) namely: $B_1 = \{1,2,4\}$, $B_2 = \{3,5,6\}$, $B_3 = \{0,2,5\}$, $B_4 = \{4,6,7\}$, $B_5 = \{0,3,4\}$, $B_6 = \{1,5,7\}$, $B_7 = \{0,1,6\}$, $B_8 = \{2,3,7\}$. For $b = 2q$, $q = 1,2,3$, we have a well balanced family by taking the blocks B_j, $1 \leq j \leq 2q$. For $b = 3$ (resp. $b = 5$) add to B_1, B_2 (resp. B_1, B_2, B_3, B_4) the block $\{0,1,7\}$. For $b = 7$ take the blocks B_j, $1 \leq j \leq 7$.

Cases $11 \leq b \leq 17$ and $b = 20$. We apply Construction C starting from the STS(7) with the 7 blocks $C_i = \{i, i+1, i+3\}$, for $0 \leq i \leq 6$ (values modulo 7) and adding a new element α. For $b = 11$, we consider the 4 covering pairs $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{0,6\}$ and add to the STS(7), the 4 blocks $E_1 = \{0,1,\alpha\}$, $E_2 = \{2,3,\alpha\}$, $E_3 = \{4,5,\alpha\}$, $E_4 = \{0,6,\alpha\}$. Note that $\lambda_x = 4$ except $\lambda_0 = 5$ and $\lambda_{x,y} = 1$ except for the covering pairs $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{0,6\}$, and $\{0,\alpha\}$. Then we can add successively $E_5 = \{1,3,5\}$, $E_6 = \{2,4,6\}$. At that point $\lambda_x = 5$, except $\lambda_\alpha = 4$ and $\lambda_{x,y} \leq 2$. We can still add $E_7 = \{1,2,\alpha\}$, $E_8 = \{0,3,4\}$, $E_9 = \{5,6,\alpha\}$, and $E_{10} = \{0,2,5\}$ getting solutions for $11 \leq b \leq 17$. For $b = 20$ we add furthermore the 3 blocks $E_{11} = \{1,4,6\}$, $E_{12} = \{3,6,\alpha\}$, $E_{13} = \{0,4,\alpha\}$. One can note that all these blocks are disjoint from those of the STS.

Cases $21 \leq b \leq 26$. We will use again Construction C, starting with the two disjoint STS(7) with blocks $C_i = \{i, i+1, i+3\}$, for $0 \leq i \leq 6$ and the second one with blocks $D_i = \{i, i+2, i+3\}$, for $0 \leq i \leq 6$ (indices modulo 7). Add the 7 blocks $F_i = \{i, i+1, \alpha\}$, $0 \leq i \leq 6$. We get a solution for $b = 21$. Note that $\lambda_x = 8$ except $\lambda_\alpha = 7$ and $\lambda_{x,y} = 2$ except for the pairs $\{i, i+1\}$ for which it is 3. Then add the blocks $\{0,4,\alpha\}$, $\{2,6,\alpha\}$, $\{1,3,5\}$ (at that point for $b = 24$, $\lambda_x = 9$) and $\{0,2,5\}$, $\{1,4,6\}$.

$\mathbf{v = 11.}$ We only need to consider $b \leq \lfloor \binom{11}{3}/2 \rfloor = 82$. By Proposition 7 there are no solutions for $b = 18, 19, 36, 37, 73, 74$. Solutions for all the other values will be constructed below.

Cases $1 \leq b \leq 10$. We take the following blocks (in given order): $\{0,1,2\}$, $\{3,4,5\}$, $\{6,7,8\}$, $\{0,9,10\}$, $\{2,5,8\}$, $\{3,6,9\}$, $\{4,7,10\}$, $\{1,5,9\}$, $\{1,8,10\}$, and $\{2,4,6\}$.

Case $b = 11$. A solution is obtained with all blocks of the form $\{i, i+1, i+3\}$ (mod 11) for $0 \leq i \leq 10$.

Cases $12 \leq b \leq 17$ and $b = 20, 21$. Solutions for $12 \leq b \leq 17$ are obtained using the results of Section 6.5. However to be complete, we give here explicit solutions. The number of edges in any $K_{11} - C_4 - C_3 - C_3$ is 55 - 10 = 45 hence a multiple of 3. The graph can therefore be decomposed into 15 K_3. For instance with $C_4 = (1,2,3,4,1)$ and $C_3 = \{5,6,9\}$ and $\{7,8,10\}$, one such decomposition is $\{0,1,7\}$, $\{0,2,5\}$, $\{0,3,10\}$, $\{0,4,9\}$, $\{0,6,8\}$, $\{1,3,6\}$, $\{1,5,10\}$, $\{1,8,9\}$, $\{2,4,8\}$, $\{2,6,7\}$, $\{2,9,10\}$, $\{3,5,8\}$, $\{3,7,9\}$, $\{4,5,7\}$, and $\{4,6,10\}$. It provides a solution for $b = 15$. Removing successively blocks $\{0,1,7\}$, $\{2,9,10\}$ and $\{3,5,8\}$ yields solutions for $b = 14$, 13 and 12. Solutions for $b = 16$ and $b = 17$ are obtained by adding back the blocks $\{5,6,9\}$ and $\{7,8,10\}$ to the solution for $b = 15$. Adding the three blocks $\{1,2,4\}$, $\{3,4,7\}$, and $\{2,3,9\}$, then block $\{0,1,5\}$, gives solutions for $b = 20$ and 21. In fact, for $b = 20$, this solution is a K_3-covering of $K_{11} + C_5$, where the 5-cycle is $(2,4,7,3,9)$.

For larger values of b, we adapt the constructions introduced in Section 6.4. We will use the two disjoint KTS(9) of Examples 1 (a) and 1 (b) as given at the beginning of section 6.4.

K_A :	$\{0,7,8\}$	$\{0,2,5\}$	$\{0,3,4\}$	$\{0,1,6\}$	K_B :	$\{1,7,8\}$	$\{1,3,6\}$	$\{1,4,5\}$	$\{0,1,2\}$
	$\{1,2,4\}$	$\{1,3,8\}$	$\{1,5,7\}$	$\{2,3,7\}$		$\{2,3,5\}$	$\{2,4,8\}$	$\{2,6,7\}$	$\{3,4,7\}$
	$\{3,5,6\}$	$\{4,6,7\}$	$\{2,6,8\}$	$\{4,5,8\}$		$\{0,4,6\}$	$\{0,5,7\}$	$\{0,3,8\}$	$\{5,6,8\}$

Note that each column forms a parallel class of the systems. Together here $b = 24$, $\lambda_x = 8$ and $\lambda_{xy} = 2$.

Cases $22 \leq b \leq 33$. We use a construction similar to Construction C. We start with K_B and add the following 10 blocks: $\{1,7,\alpha\}$, $\{1,8,\beta\}$, $\{2,3,\alpha\}$, $\{2,5,\beta\}$, $\{0,4,\alpha\}$, $\{0,6,\beta\}$, $\{5,6,\alpha\}$, $\{3,4,\beta\}$, $\{\alpha,\beta,7\}$ and $\{\alpha,\beta,8\}$. That gives a solution for $b = 22$. Here $\lambda_x = 6$ and $\lambda_{x,y} = 1$ or 2 (11 pairs). We use the solution to construct solutions for some other values of b: (i) adding the block(s) $\{0,1,5\}$, $\{2,4,7\}$, and $\{3,6,8\}$ gives the solutions for $b = 23, 24$, and 25. (ii) adding the 4 blocks $\{0,1,5\}$, $\{2,4,7\}$, $\{6,8,\alpha\}$, and $\{3,5,\beta\}$ gives a solution for $b = 26$, and then adding $\{1,4,6\}$ or $\{0,2,8\}$ results solutions for $b = 27, 28$.

Consider the solution for $b = 25$: (i) adding the 4 blocks $\{4,6,\alpha\}$, $\{5,8,\alpha\}$, $\{1,2,\beta\}$, $\{0,3,\beta\}$, we have a solution for $b = 29$; then adding $\{3,5,7\}$ we obtain a solution for $b = 30$, and adding $\{0,7,8\}$ a solution for $b = 31$. (ii) adding the 7 blocks: $\{0,7,8\}$, $\{1,2,\alpha\}$, $\{4,6,\alpha\}$, $\{3,5,\alpha\}$, $\{1,6,\beta\}$, $\{0,2,\beta\}$, and $\{3,7,\beta\}$ gives a solution for $b = 32$. Adding the block $\{4,5,8\}$, we get a solution for $b = 33$.

Cases $33 \leq b \leq 35$ and $38 \leq b \leq 44$. We use a construction similar to Construction A. We add two new vertices α and β and replace each block $\{x,y,z\}$ in the first parallel class of K_A by three blocks: $\{x,y,\alpha\}$, $\{x,z,\alpha\}$, and $\{y,z,\alpha\}$, and repeat this operation to the second parallel class of K_A with β. There are $b = 36$ blocks in total now. Now, $\lambda_x = 10$, $\lambda_\alpha = \lambda_\beta = 9$, and $\lambda_{xy} = \lambda_{x\beta} = \lambda_{x\alpha} = 2$ and $\lambda_{\alpha\beta} = 0$. This solution is of course not well balanced (as no well balanced design exits for $b = 36$) , but we will use it to construct solutions for $33 \leq b \leq 35$ and $38 \leq b \leq 44$: (i) delete the two blocks $\{0,7,\alpha\}$ and $\{0,2,\beta\}$, and add the block $\{0,\alpha,\beta\}$. Now, $\lambda_{x,y} = 2$ except $\lambda_{0,7}, \lambda_{0,2}, \lambda_{7,\alpha}, \lambda_{2,\beta}, \lambda_{\alpha,\beta} = 1$, and $\lambda_x = 10$ except λ_0, λ_2, λ_7, $\lambda_\alpha, \lambda_\beta = 9$. This gives a solution for $b = 35$. Furthermore, deleting block $\{1,3,6\}$ and then $\{4,5,8\}$ from the solution for $b = 35$ gives solutions for $b = 34, 33$. (ii) adding two blocks $\{\alpha,\beta,0\}$ and $\{\alpha,\beta,3\}$, we have a solution for $b = 38$. Here $\lambda_x = 10$ except $\lambda_0, \lambda_3, \lambda_\alpha, \lambda_\beta = 11$ and $\lambda_{x,y} = 2$ except

$\lambda_{0,\alpha}, \lambda_{0,\beta}, \lambda_{3,\alpha}, \lambda_{3,\beta} = 3$. Now adding blocks $\{1,2,5\}$, $\{4,6,8\}$, $\{3,6,7\}$, $\{0,1,8\}$, $\{2,4,7\}$, and $\{\alpha,\beta,5\}$ we have the solutions for $38 \leq b \leq 44$.

Cases $45 \leq b \leq 72$ and $b = 75, 76$. The solution for $b = 55$ can be obtained from a $(11,3,3)$-design. Here $\lambda_{xy} = 3$ and $\lambda_x = 15$. A solution consists of the 5 classes $\{i, i+1, i+2\}, \{i, i+2, i+4\}, \{i, i+3, i+6\}, \{i, i+4, i+8\}$ $\{i, i+5, i+10\}$ (the values are taken modulo 11).

Let us now introduce a device which is useful to quickly identify pairs in proposed solutions. To a given block $\{a,b,c\}$ we associate its "difference family", the (unordered) list made of the three "smallest" differences between values of a block (a pair $\{a,b\}$ has two possible differences $a-b$ and $b-a$ modulo 11). Note that all the blocks of the class obtained by translating a given block, that is the blocks $\{a+i, b+i, c+i\}$ (values are taken modulo 11), have the same difference family. The converse is not true; for example blocks with difference family 123 can be in the class $\{i, i+1, i+3\}$ or $\{i, i+2, i+3\}$.

The solution above is then generated by the 5 difference families: 112, 224, 335, 443, 551. Note that each difference occurs three times. This solution is now used to obtain solutions for the following values of b.

(i) for $45 \leq b \leq 54$, just deleting some or all of the 10 blocks for $b = 10$ (note that the difference families of these blocks are in the following set: 112, 224, 335, and 443),

(ii) adding the following 10 blocks gives solutions for $56 \leq b \leq 65$: $\{0,1,3\}$, $\{4,5,7\}$, $\{2,8,10\}$, $\{5,6,9\}$, $\{3,4,6\}$, $\{0,7,10\}$, $\{1,2,9\}$, $\{0,8,9\}$, $\{2,3,5\}$, and $\{1,7,8\}$ as all these blocks have difference families with no repetition,

(iii) adding a class of 11 blocks with difference family 123, for example the blocks $\{i, i+1, i+3\}$ gives a solution for $b = 66$.

Finally, observe that the blocks in the solutions for $12 \leq b \leq 17$ and $b = 20, 21$, have difference families different from $ii(2i)$ $(1 \leq i \leq 5)$, whereas in the solution for $b = 55$, only blocks with difference families $ii(2i)$ are used. Therefore, combining the above solutions with $b = 55$, we have solutions for $67 \leq b \leq 72$ and $b = 75, 76$.

Cases $77 \leq b \leq 82$. For $b = 77$, take the following 7 classes: $(i, i+1, i+2)$, $(i, i+2, i+4)$, $(i, i+3, i+6)$, $(i, i+4, i+8)$, $(i, i+5, i+10)$, $(i, i+1, i+3)$, $(i, i+1, i+5)$. The corresponding difference families are $112, 224, 335, 443, 551, 123, 145$. Hence $\lambda_x = 21$, $\lambda_{xy} = 5$ for pairs with difference 1, and 4 for all the pairs with other differences. Now adding some or all the blocks: $\{0,2,5\}$, $\{1,3,7\}$, $\{4,6,9\}$, $\{2,8,10\}$, $\{3,5,8\}$ gives solution for $78 \leq b \leq 82$.

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

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