Tutorial on Control and State Constrained Optimal Control Problems

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Tutorial on Control and State Constrained Optimal Control Problems – Part 2 : Mixed Control-State Constraints

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Outline

Optimal Control Problem

state $x(t) \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^m$.

Dynamics and Boundary Conditions

$$\dot{x}(t) = f(t, x(t), u(t)), \ \text{a.e. } t \in [0, t_f],$$
$$x(t) = \bar{x}_0 \,, \ \psi(x(t_f)) = 0 \quad (\psi : \mathbb{R}^n \to \mathbb{R}^r).$$

Control and State Constraints

$$\begin{array}{llll} c(x(t), u(t)) & \leq 0\,, & 0 \leq t \leq t_f\,, & (c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k) \\ s(x(t)) & \leq 0\,, & 0 \leq t \leq t_f\,, & (s : \mathbb{R}^n \to \mathbb{R}^l) \end{array}$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(t, x(t), u(t))\, dt$$

Discretization

For simplicity consider a MAYER–type problem with cost functional

$$J(u,x) = g(x(t_f)) \,.$$

This can achieved by considering the additional state variable x_0 with

$$\dot{x}_0 = f_0(x,u)\,, \quad x_0(0) = 0\,.$$

Then we have

$$x_0(t_f) = \int_0^{t_f} f_0(t, x(t), u(t))\,.$$

Choose an integer $N \in \mathbb{N}$, a stepsize h and grid points t_i :

$$h = t_f/N\,, \quad t_i := ih\,, \quad (i = 0, 1, \ldots, N)\,.$$

Approximation of control and state at grid points:

$$u(t_i) \approx u_i \in \mathbb{R}^m\,, \quad x(t_i) \approx x_i \in \mathbb{R}^n \quad (i = 0, \ldots, N)$$

Large-scale NLP using EULER's method

Minimize

$$J(u,x) = g(x_N)$$

subject to

$$\begin{aligned} x_{i+1} &= x_i + h \cdot f(t_i, x_i, u_i), \quad & i = 0,..,N-1, \\ x_0 &= \bar{x}_0\,,\ \psi(x_N) = 0, & \\ c(x_i, u_i) &\le 0\,, & i = 0,..,N, \\ s(x_i) &\le 0\,, & i = 0,..,N, \end{aligned}$$

Optimization variable for full discretization:

$$z := (u_0, x_1, u_1, x_2, ..., u_{N-1}, x_N, u_N) \in \mathbb{R}^{N(m+n)+m}$$

NLP Solvers

- AMPL : Programming language (Fourer, Gay, Kernighan)
- IPOPT : Interior point method (Andreas Wächter)
- LOQO : Interior point method (Vanderbei et al.)
- Other NLP solvers embedded in AMPL : cf. NEOS server
- NUDOCCCS : optimal control package (Christof Büskens)
- WORHP : SQP solver (Christof Büskens, Matthias Gerdts)
- Special feature: solvers provide LAGRANGE-multipliers as approximations of the adjoint variables.

Optimal Control Problem with Control-State Constraints

State $x(t) \in \mathbb{R}^n$, Control $u(t) \in \mathbb{R}^m$.
All functions are assumed to be suffciently smooth

Dynamics and Boundary Conditions

$$\dot{x}(t) = f(x(t), u(t)), \text{ a.e. } t \in [0, t_f],$$
$$x(0) = x_0 \in \mathbb{R}^n, \ \psi(x(t_f)) = 0 \in \mathbb{R}^k,$$
$$(\, 0 = \varphi(x(0), x(t_f)) \quad \text{mixed boundary conditions} \,)$$

Mixed Control-State Constraints

$$\alpha \leq c(x(t), u(t)) \leq \beta, \quad t \in [0, t_f], \quad c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

Control bounds $\alpha \leq u(t) \leq \beta$ are included by $c(x, u) = u$.

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} f_0(x(t), u)\, dt$$

Hamiltonian

Hamiltonian

$$H(x,\lambda,u) = \lambda_0\, f(x,u) + \lambda\, f(x,u) \qquad \lambda \in \mathbb{R}^n \text{ (row vector)}$$

Augmented Hamiltonian

$$\begin{aligned} \mathcal{H}(x,\lambda,\mu,u) &= H(x,\lambda,u) + \mu\, c(x,u) \\ &= \lambda_0\, f(x,u) + \lambda\, f(x,u) + \mu\, c(x,u), \qquad \mu \in \mathbb{R}\,. \end{aligned}$$

Let $(u,x) \in \mathcal{L}^\infty([0,T],\mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0,T],\mathbb{R}^n)$ be a locally optimal pair of functions.

Regularity assumption

$$c_u(x(t),u(t)) \neq 0 \quad \forall\, t \in J_a$$

$$J_a := \{\, t \in [0,t_f] \mid c(x(t),u(t)) = \alpha \ \text{ or } \ = \beta \,\}$$

Minimum Principle of Pontryagin et al. and Hestenes

Let $(u, x) \in \mathcal{L}^\infty([0, t_f], \mathbb{R}^m) \times \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ be a locally optimal pair of functions that satisfies the regularity assumption. Then there exist

- an adjoint (costate) function $\lambda \in \mathcal{W}^{1,\infty}([0, t_f], \mathbb{R}^n)$ and a scalar $\lambda_0 \geq 0$,
- a multiplier function $\mu \in \mathcal{L}^\infty([0, t_f], \mathbb{R})$,
- and a multiplier $\rho \in \mathbb{R}^r$ associated to the boundary condition $\psi(x(t_f)) = 0$

that satisfy the following conditions for a.a. $t \in [0, t_f]$, where the argument (t) denotes evaluation along the trajectory $(x(t), u(t), \lambda(t))$:

Minimum Principle of Pontraygin et al. and Hestenes

(i) Adjoint ODE and transversality condition:

$$\begin{aligned} \dot{\lambda}(t) &= -\mathcal{H}_x(t) = -(\lambda_0 f_0 + \lambda f)_x(t) - \mu(t)\, c_x(t)\,, \\ \lambda(t_f) &= (\lambda_0\, g + \rho\psi)_x(x(t_f))\,, \end{aligned}$$

(iia) Minimum Condition for Hamiltonian:

$$H(x(t), \lambda(t), u(t)) = min\,\{\, H(x(t), \lambda(t), u) \mid \alpha \le c(x(t), u) \le \beta \,\}$$

(iib) Local Minimum Condition for Augmented Hamiltonian:

$$0 = \mathcal{H}_u(t) = (\lambda_0 f_0 + \lambda f)_u(t) + \mu(t)\, c_u(t)$$

(iii) Sign of multiplier μ and complementarity condition:

$$\begin{aligned} &\mu(t) \le 0, \text{ if } c(x(t), u(t)) = \alpha\,;\ \mu(t) \ge 0, \text{ if } c(x(t), u(t)) = \beta\,, \\ &\mu(t) = 0\,, \text{ if } \alpha < c(x(t), u(t)) < \beta\,. \end{aligned}$$

Evaluation of the Minimum Principle: boundary arc

Boundary arc: Let $[t_1, t_2]$, $0 \le t_1 < t_2 < t_f$, be an interval with

$$c(x(t), u(t)) = \alpha \quad \text{or} \quad c(x(t), u(t)) = \beta \quad \forall\, t_1 \le t \le t_2 .$$

For simplicity assume a scalar control, i.e., $m = 1$.
Due to the regularity condition $c_u(x(t), u(t)) \neq 0$ there exists a smooth function $u_b(x)$ satisfying

$$c(x, u_b(x)) \equiv \alpha \; (\equiv \beta) \quad \forall\, x \text{ in a neighborhood of the trajectory.}$$

The control $u_b(x)$ is called the boundary control and yields the optimal control by the relation $\boxed{u(t) = u_b(x(t))\,.}$

It follows from the local minimum condition $0 = \mathcal{H}_u = H_u + \mu\, c_u$ that the multiplier μ is given by

$$\boxed{\mu = \mu(x, \lambda) = -H_u(x, \lambda, u_b(x)) \,/\, c_u(x, u_b(x))\,.}$$

Case I : Regular Hamiltonian, u is continuous

CASE I : Consider optimal control problems which satisfy the

Assumption: The Hamiltonian $H(x, \lambda, u)$ is regular, i.e., it admits a unique minimum u. The strict Legendre condition holds:

$$H_{uu}(t) > 0 \quad \forall\, t \in [0, t_f]\,.$$

(a) Then there exists a "free control" $u = u_{free}(x, \lambda)$ satisfying

$$H_u(x, \lambda, u_{free}(x, \lambda)) \equiv 0\,.$$

(b) The optimal control $u(t)$ is continuous in $[0, t_f]$.

Claim (b) follows from the continuity and regularity of H.

The continuity of the control implies junctions conditions at junction points t_k $(k = 1, 2)$ with the boundary:

$$\boxed{u_{free}(x(t_k), \lambda(t_k)) = u_b(x(t_k))\,, \quad \mu(t_k) = 0 \quad (k = 1, 2).}$$

Rayleigh Problem with Quadratic Control

The Rayleigh problem is a variant of the van der Pol Oszillator, where x_1 denotes the electric current.

Control problem for the Rayleigh Equation

Minimize $J(x,u) = \int_0^{t_f} (u^2 + x_1^2)\, dt \quad (t_f = 4.5)$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14 x_2^2) + 4u, & x_2(0) &= -5. \end{aligned}$$

Three types of constraints:

Case (a) : no control constraints.
Case (b) : control constraint $-1 \le u(t) \le 1$.
Case (c) : mixed control-state constraint

$$\alpha \le u(t) + x_1(t)/6 \le 0, \ \alpha = -1, -2$$

Case I (a) : Rayleigh problem, no constraint

Normal Hamiltonian:

$$H(x,\lambda,u) = u^2 + x_1^2 + \lambda_1\, x_2 + \lambda_2\,(-x_1 + x_2(1.4 - 0.14x_2^2) + 4u)$$

Adjoint Equations:

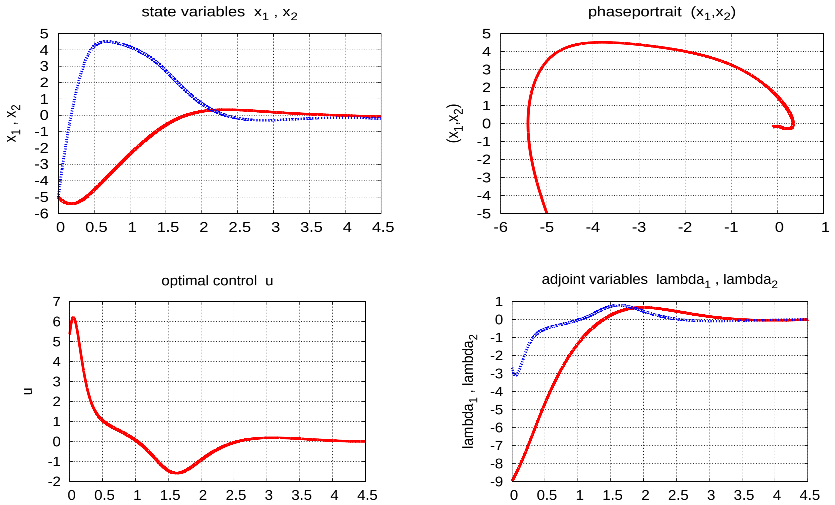
$$\begin{aligned} \dot\lambda_1 &= -H_{x_1} = -2x_1 + \lambda_2 & \lambda_1(t_f) = 0, \\ \dot\lambda_2 &= -H_{x_2} = -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) & \lambda_2(t_f) = 0, \end{aligned}$$

Minimum condition:

$$0 = H_u = 2u + 4\lambda_2 \quad \Rightarrow \quad u = u_{free}(x,\lambda) = -2\lambda_2\,.$$

Shooting method for solving the boundary value problem for (x,λ):
Determine unknown shooting vector $s = \lambda(0) \in \mathbb{R}^2$ that satisfies the terminal condition $\lambda(t_f) = 0$: use Newton's method

Case I : Rayleigh problem without constraints



Note: Hamiltionian is regular, control $u(t)$ is continuous (analytic).

Case I (b) : Rayleigh problem, constraint $-1 \le u(t) \le 1$

Hamiltonian H and adjoint equations are as in Case (a).
The free control is given by $u_{free}(x, \lambda) = -2\lambda_2$.
Structure of optimal control:

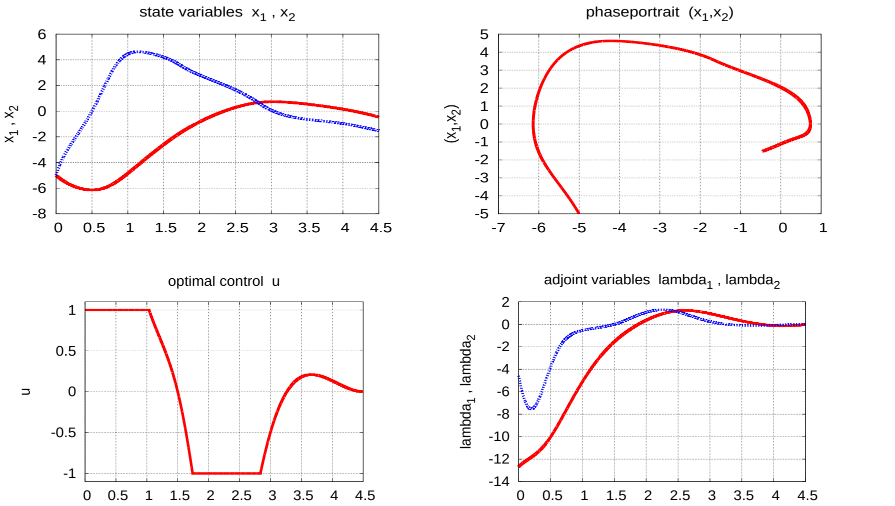
$$u(t) = \left\{ \begin{array}{rll} 1 & \text{for} & 0 \le t \le t_1 \\ -2\lambda_2(t) & \text{for} & t_1 \le t \le t_2 \\ -1 & \text{for} & t_2 \le t \le t_3 \\ -2\lambda_2(t) & \text{for} & t_3 \le t \le t_f \end{array} \right\}$$

Junction conditions: Continuity of the control implies

$$u(t_k) = -2\lambda_2(t_k) = 1 \mid -1 \mid -1\,, \quad k = 1, 2, 3\,.$$

Shooting method for solving the boundary value problem for (x, λ):
Determine shooting vector $s = (\lambda(0), t_1, t_2, t_3) \in \mathbb{R}^{2+3}$ that satisfies 2 terminal conditions $\lambda(t_f) = 0$ and 3 junction conditions.

Rayleigh problem with control constraint $|u(t)| \le 1$



Note: Hamiltonian is regular, control $u(t)$ is continuous.

Junction conditions: $-2\lambda_2(t_k) = 1 \mid -1 \mid -1, \quad k = 1, 2, 3$

Case I (c) : Rayleigh problem, $-1 \le u + x_1/6 \le 0$

Augmented (normal) Hamiltonian:

$$\begin{aligned} \mathcal{H}(x,\lambda,\mu,u) &= u^2 + x_1^2 + \lambda_1 x_2 \\ &\quad + \lambda_2 \left(-x_1 + x_2(1.4 - 0.14x_2^2) + 4u\right) + \mu(u + x_1/6) \end{aligned}$$

Adjoint Equations:

$$\begin{aligned} \dot\lambda_1 &= -\mathcal{H}_{x_1} = -2x_1 + \lambda_2 - \mu/6\,, & \lambda_1(t_f) &= 0, \\ \dot\lambda_2 &= -H_{x_2} = -\lambda_1 - \lambda_2(1.4 - 0.42x_2^2) & \lambda_2(t_f) &= 0, \end{aligned}$$

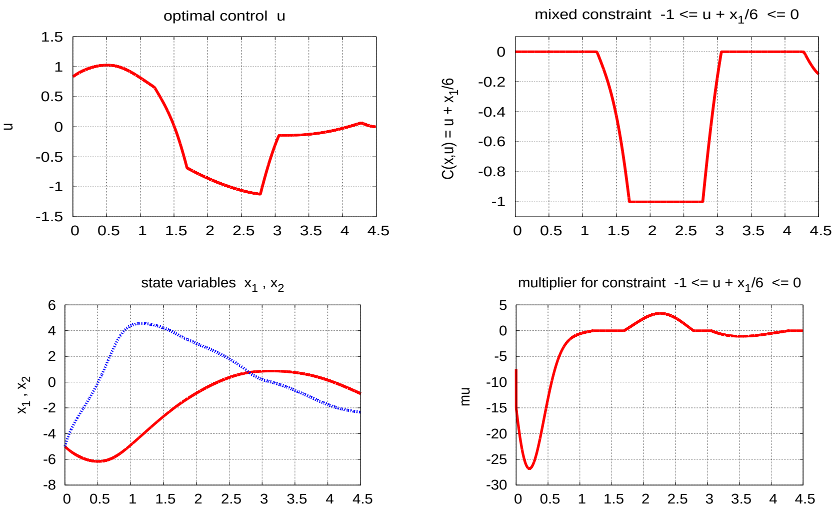
Free control : $u_{free}(x,\lambda) = -2\lambda_2\,.$

Boundary control : $u_b(x) = \alpha - x_1/6 \quad \text{for} \quad \alpha \in \{-1, 0\}\,.$

Multiplier :

$$\mu = \mu(x,\lambda) = -H_u(x,\lambda,u_b(x))\,/\,c_u(x,u_b(x)) = 2u_b(x) + 4\lambda_2\,.$$

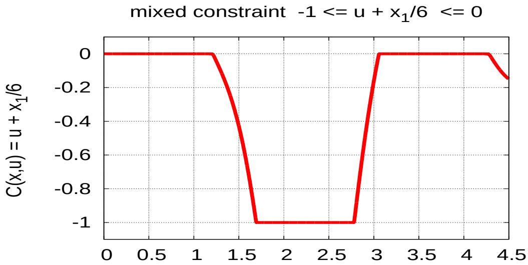
Case I (c) : Rayleigh problem, $-1 \le u + x_1/6 \le 0$



Hamiltonian is regular, control $u(t)$ is continuous.

Junction conditions: $-2\lambda(t_k) = \alpha - x_1(t_k)/6\,,\, \alpha \in \{0, -1\}$.

Case I (c) : Rayleigh problem, structure of optimal control for mixed constraint $-1 \le u + x_1/6 \le 0$



$$u(t) = \left\{ \begin{array}{rll} -x_1/6 & \text{for} & 0 \le t \le t_1 \\ -2\lambda_2(t) & \text{for} & t_1 \le t \le t_2 \\ -1 - x_1/6 & \text{for} & t_2 \le t \le t_3 \\ -2\lambda_2(t) & \text{for} & t_3 \le t \le t_4 \\ -x_1/6 & \text{for} & t_4 \le t \le t_5 \\ -2\lambda_2(t) & \text{for} & t_5 \le t \le t_f \end{array} \right\}$$

Case II : control u appears linearly

CASE II : Control appears linearly in the cost functional, dynamics and mixed control-state constraint. Let u be scalar.

Dynamics and Boundary Conditions

$$\dot{x}(t) = f_1(x(t)) + f_2(x(t)) \cdot u(t), \text{ a.e. } t \in [0, t_f],$$
$$x(0) = x_0 \in \mathbb{R}^n, \quad \psi(x(t_f)) = 0 \in \mathbb{R}^k ,$$

Mixed Control-State Constraints

$$\alpha \le c_1(x(t)) + c_2(x(t)) \cdot u(t) \le \beta \quad \forall t \in [0, t_f]. \quad c_1, c_2 : \mathbb{R}^n \to \mathbb{R}$$

Minimize

$$J(u, x) = g(x(t_f)) + \int_0^{t_f} (\, f_{01}(x(t)) + f_{02}(x(t)\,) \cdot u(t)\, dt$$

Case II : Hamiltonian and switching function

Normal Hamiltonian

$$H(x, \lambda, u) = f_{01}(x) + \lambda f_1(x) + [\, f_{02}(x) + \lambda f_2(x)\,] \cdot u\,.$$

Augmented Hamiltonian

$$\mathcal{H}(x, \lambda, \mu, u) = H(x, \lambda, u) + \mu\,(c_1(x) + c_2(x) \cdot u)$$

The optimal control $u(t)$ solves the minimization problem

$$\min \{\, H(x(t), \lambda(t), u) \mid \alpha \le c_1(x(t)) + c_2(x(t)) \cdot u \le \beta \,\}$$

Define the switching function

$$\sigma(x, \lambda) = H_u(x, \lambda, u) = f_{02}(x) + \lambda f_2(x)\,, \quad \sigma(t) = \sigma(x(t), \lambda(t))\,.$$

Case II : Hamiltonian

The minimum condition is equivalent to the minimization problem

$$\min\{\,\sigma(t)\cdot u)\mid \alpha \le c_1(x(t)) + c_2(x(t))\cdot u \le \beta\,\}$$

We deduce the control law

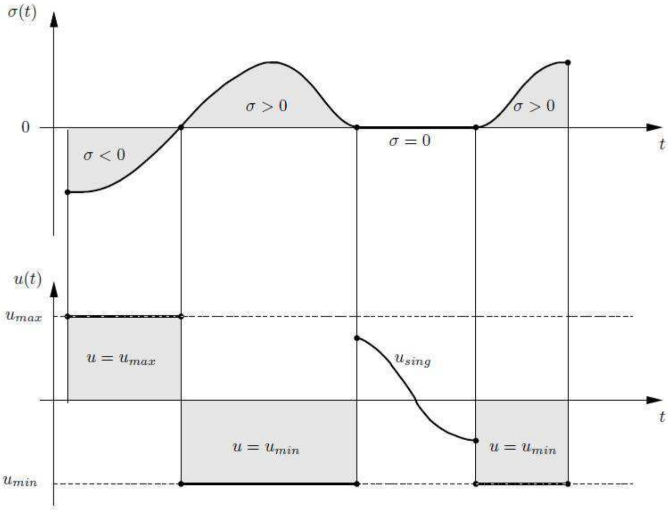
$$c_1(x(t))+c_2(x(t))\cdot u(t) = \left\{ \begin{array}{cll} \alpha\,, & \text{if} & \sigma(t)\cdot c_2(x(t)) > 0 \\ \beta\,, & \text{if} & \sigma(t)\cdot c_2(x(t)) < 0 \\ \text{undetermined} & \text{if} & \sigma(t) \equiv 0 \end{array} \right\}$$

The control u is called bang-bang in an interval $I \subset [0, t_f]$, if $\sigma(t)\cdot c_2(x(t)) \neq 0$ for all $t \in I$. The control u is called singular in an interval $I_{\rm sing} \subset [0, t_f]$, if $\sigma(t)\cdot c_2(x(t)) \equiv 0$ for all $t \in I_{\rm sing}$.

For the control constraint $\alpha \le u(t) \le \beta$ with $c_1(x) = 0,\ c_2(x) = 1$ we get the classical control law

$$u(t) = \left\{ \begin{array}{cll} \alpha\,, & \text{if} & \sigma(t) > 0 \\ \beta\,, & \text{if} & \sigma(t) < 0 \\ \text{undetermined}\,, & \text{if} & \sigma(t) \equiv 0 \end{array} \right\}$$

Bang-Bang and Singular Controls



Case II : Rayleigh problem with $-1 \le u(t) \le 1$

Rayleigh problem with control appearing linearly

Minimize $\quad J(x,u) = \int\limits_0^{t_f} (x_1^2 + x_2^2)\, dt \quad (t_f = 4.5)$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5\,, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14 x_2^2) + 4u\,, & x_2(0) &= -5, \\ -1 &\le u(t) \le 1\,. \end{aligned}$$

Adjoint Equations:

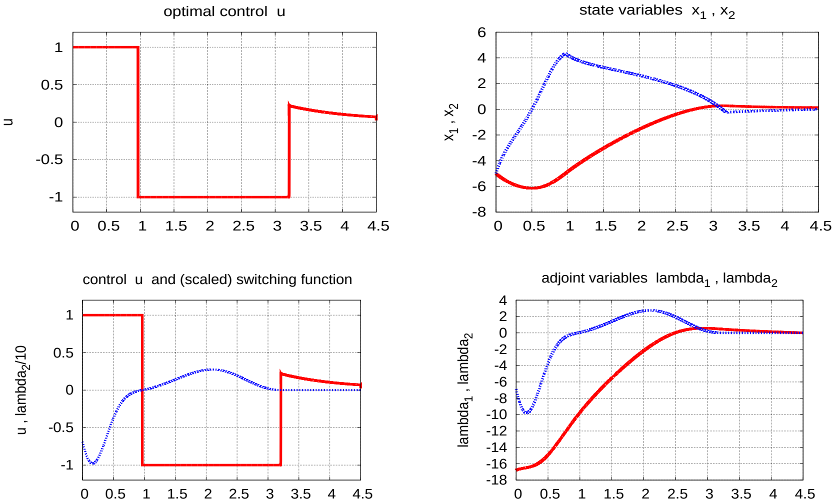
$$\begin{aligned} \dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2\,, & \lambda_1(t_f) &= 0, \\ \dot{\lambda}_2 &= -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1.4 - 0.42 x_2^2)\,, & \lambda_2(t_f) &= 0, \end{aligned}$$

The switching function $\sigma(t) = H_u(t) = 4\lambda_2(t)$ gives the control law

$$\boxed{u(t) = -\mathrm{sign}\,(\lambda_2(t))}$$

.

Case II : Rayleigh problem, $-1 \le u(t) \le 1$



Control $u(t)$ is bang-bang-singular.

Switching conditions: $\lambda(t_1) = 0\,,\;\; \lambda_2(t) \equiv 0 \;\forall\; t \in [t_2, t_f]$.

Case II : Rayleigh problem, $\alpha \le u + x_1/6 \le 0$

Minimize $J(x,u) = \int_0^{t_f} (x_1^2 + x_2^2)\,dt \quad (t_f = 4.5)$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2, & x_1(0) &= -5\,, \\ \dot{x}_2 &= -x_1 + x_2(1.4 - 0.14x_2^2) + 4u\,, & x_2(0) &= -5, \end{aligned}$$

and the mixed control-state constraint

$$\alpha \le u(t) + x_1(t)/6 \le 0 \quad \forall\, 0 \le t \le t_f\,.$$

Adjoint Equations:

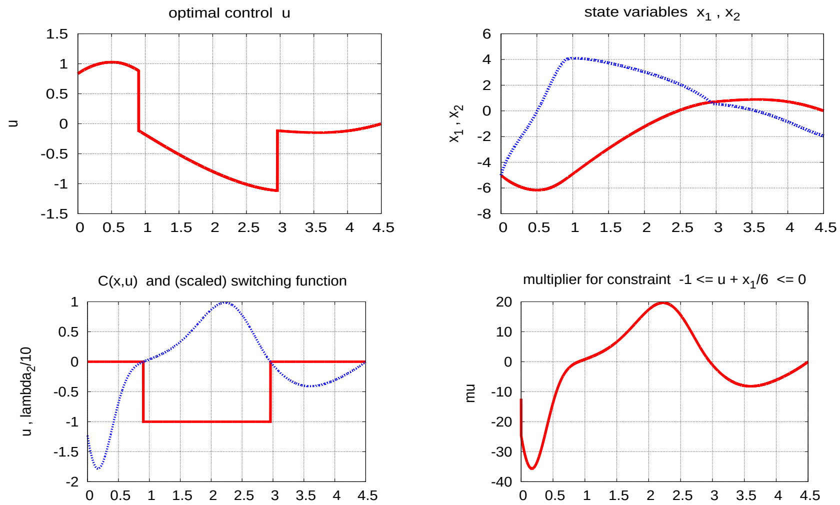
$$\begin{aligned} \dot{\lambda}_1 &= -H_{x_1} = -2x_1 + \lambda_2 - \mu/6\,, & \lambda_1(t_f) &= 0, \\ \dot{\lambda}_2 &= -H_{x_2} = -2x_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2)\,, & \lambda_2(t_f) &= 0, \end{aligned}$$

Control law for $\alpha \leq u + x_1/6 \leq 0$

The switching function is $\sigma(t) = H_u(t) = 4\lambda_2(t)$.
In view of $c_2(x) \equiv 1$ we have the control law

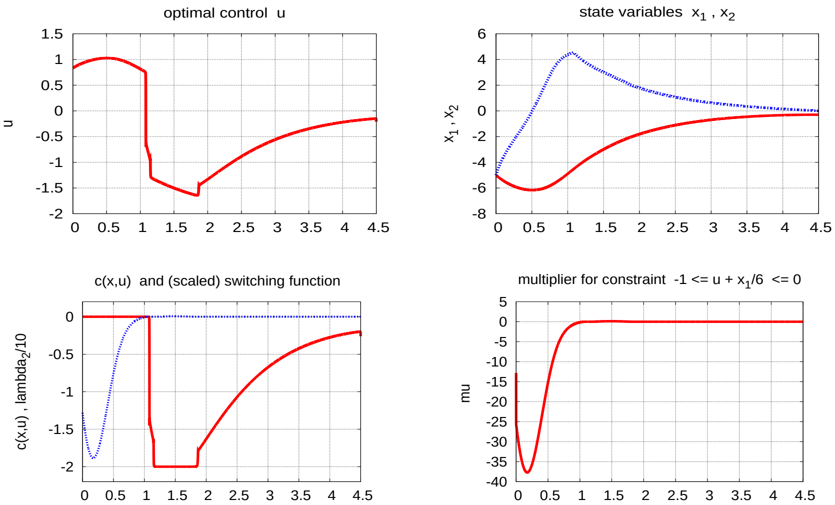
$$u + x_1/6 = \left\{ \begin{array}{lll} \alpha < 0 & , & \text{if} \quad \lambda_2(t) > 0 \\ 0 & , & \text{if} \quad \lambda_2(t) < 0 \\ \text{undetermined} & , & \text{if} \quad \lambda_2(t) \equiv 0 \end{array} \right\}$$

Case II : Rayleigh problem, $-1 \leq u + x_1/6 \leq 0$



Note: constraint $u(t) + x_1(t)/6$ is "bang-bang".

Case II : Rayleigh problem, $-2 \leq u + x_1/6 \leq 0$



Note: constraint $u(t) + x_1(t)/6$ is "bang-singular-bang-singular".

Optimal Fishing, Clark, Clarke, Munro

Colin W. Clark, Frank H. Clarke, Gordon R. Munro: *The optimal exploutation of renewable resource stock: problem of irreversible investment*, Econometric **47**, pp. 25–47 (1979).

State variables and control variables:

$x(t)$	:	population biomass at time $t \in [0, t_f]$, renewable resource, e.g., fish,
$K(t)$	:	amount of capital invested in the fishery, e.g., number of "standardized" fishing vessels available,
$E(t)$	:	fishing effort (control), $h(t) = E(t)x(t)$ is harvest rate ,
$I(t)$	:	investment rate (control),

Optimal Fishing: optimal control model

Dynamics in $[0, t_f]$ (here: $a = 1$, $b = 5$, $\gamma = 0$)

$$\dot{x}(t) = a \cdot x(t) \cdot (1 - x(t)/b) - E(t) \cdot x(t)\,, \qquad x(0) = x_0\,,$$
$$\dot{K}(t) = I(t) - \gamma \cdot K(t)\,, \qquad K(0) = K_0\,.$$

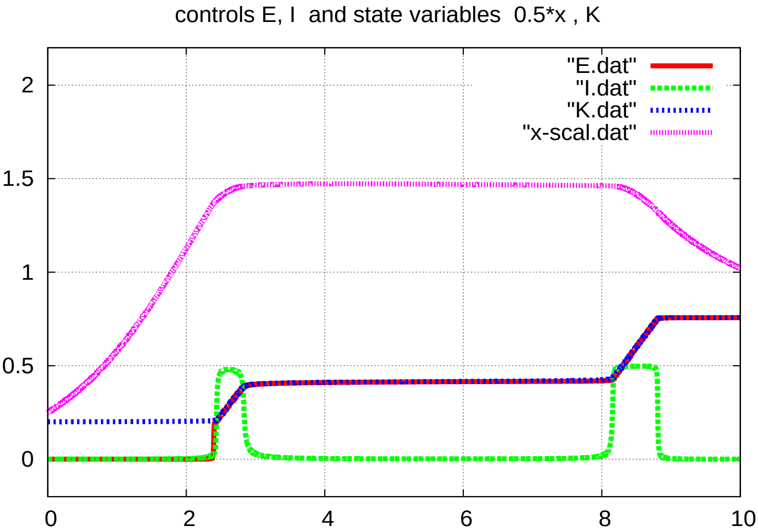
Mixed Control-State Constraint and Control Constraint

$$0 \le E(t) \le K(t)\,, \quad 0 \le I(t) \le I_{max}\,, \quad t \in [0, t_f],$$

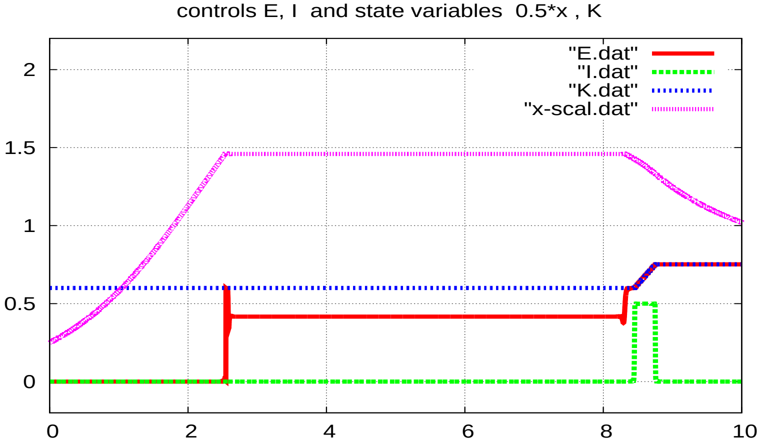
Maximize benefit (parameters: $r = 0.05$, $c_E = 2$, $c_I = 1.1$)

$$J(u, x) = \int_0^{t_f} exp(-r \cdot t)(\, p \cdot E(t) \cdot x(t) - c_E \cdot E(t) - c_I \cdot I(t)\,)\, dt$$

Optimal Fishing: $x_0 = 0.5,\ K_0 = 0.2,\ I_{max} = 0.5$



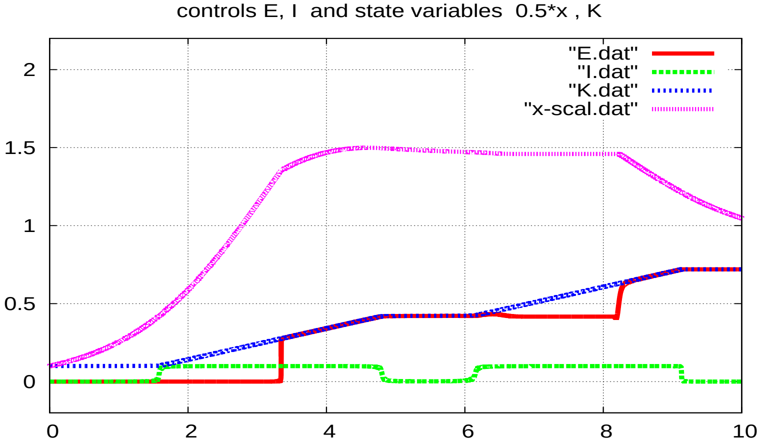
Optimal Fishing: $x_0 = 0.5$, $K_0 = 0.6$, $I_{max} = 0.5$



Fishing rate : $E(t) = 0$, $E(t) =$ singular, $E(t) = K(t)$.

Investment rate : $I(t) = 0$, $I(t) = I_{max}$, $I(t) = 0$.

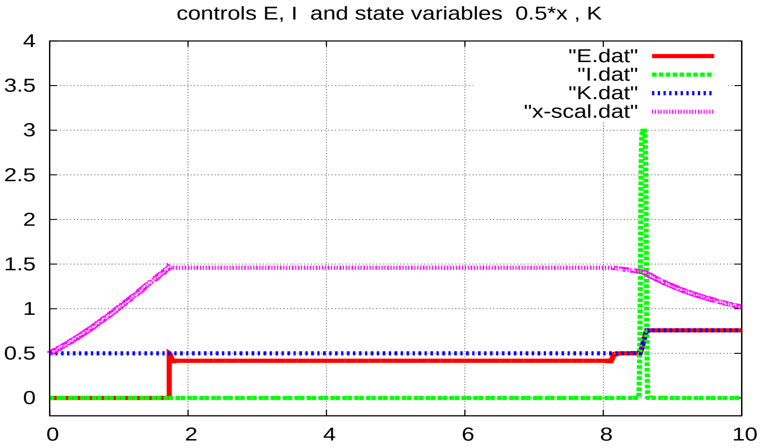
Optimal Fishing: $x_0 = 0.2$, $K_0 = 0.1$, $I_{max} = 0.1$



Fishing rate : $E(t) = 0$, $E(t) = K(t)$, $E(t)$ singular, $E(t) = K(t)$.

Investment rate : 2 arcs with $I(t) = I_{max}$.

Optimal Fishing: $x_0 = 1.0,\ K_0 = 0.5,\ I_{max} = 3$



Fishing rate : $E(t) = 0,\ E(t)$ singular $E(t) = K(t)$,

Investment rate : 1 "impulse" with $I(t) = I_{max}$.