Tutorial on Differential Games

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Differential Games III

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Contents

The differential Game

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)), \; t \in [0, T] \\ u(t) \in U, \; v(t) \in V \end{cases} \tag{1}$$

where $f : IR^N \times U \times V \to IR^N$**,** U **and** V **being the control sets of the players. To any initial condition** $x(t_0) = x_0$ **we associate** $t \to X_t^{t_0,x_0,u,v}$ **the solution to (1).**

The first player—choosing u**—wants to minimize a final cost of the form**

$$g(x(T))$$

while the second player,—playing with v**—wants to maximize it.**

the state-space x_0 is only unperfectly known by the players : they only know that the initial position is randomly distributed under some fixed probability measure μ_0. Both players are assumed to know this probability μ_0, and have a perfect knowledge of the control of the other player.

So the "lack" of information is very specific :

- it is symmetric for both player
- it is only concerned with the current position of the game.

We denote by $\mathcal{M}$ the Borel probability measures μ s.t.

$$\int_{IR^N} |x|^2 d\mu(x) < +\infty \; .$$

Assumptions

$$
\left\{
\begin{array}{ll}
(i) & U \text{ and } V \text{ are compact subsets of some finite} \\
 & \text{dimensional spaces} \\
(ii) & f : I\!R^N \times U \times V \to I\!R^N \text{ is continuous and} \\
 & \text{Lipschitz continuous with respect to} \\
(iii) & \forall (x,u,v),\ |f(x,u,v)| \le M \\
(iv) & g : I\!R^N \to I\!R \text{ is Lipschitz continuous and bounded}
\end{array}
\right. \tag{2}
$$

$$\mathcal{U}(t_0) = \{u : [t_0, T] \to U,\ \text{Lebesgue measurable}\}$$

$$\mathcal{V}(t_0) = \{v : [t_0, T] \to V,\ \text{Lebesgue measurable}\}$$

Strategies

Definition 1 ***A NAD strategy is a map*** $\alpha : \mathcal{V}(t_0) \to \mathcal{U}(t_0)$ ***such that there is some*** $\tau > 0$ ***such that*** $\forall t$***,***
$\forall v_1, v_2 \in \mathcal{V}(t_0)$***,*** $v_1 = v_2$ ***on*** $[t_0, t] \Rightarrow \alpha(v_1) = \alpha(v_2)$ ***on*** $[t_0, t+\tau]$***.***

$\mathcal{A}(t_0)$ ***is the set of such*** α***. Symmetrically,*** $\mathcal{B}(t_0)$ ***is the set of NAD strategies*** $\beta : \mathcal{U}(t_0) \to \mathcal{V}(t_0)$ ***for the second player.***

Lemma 2 $\forall (\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$***,*** $\exists! (u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$***,***

$$\alpha(v_0) = u_0 \text{ and } \beta(u_0) = v_0 \ .$$

$$X_t^{t_0,x,\alpha,\beta} := X_t^{t_0,x,u_0,v_0} \qquad \forall t \in [t_0, T] \ .$$

$g : IR^N \mapsto IR$ **which is Lipschitz and bounded. For any** $(t_0, \mu_0) \in [0, T) \times \mathcal{M}$ **and for any** $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ **we set**

$$J(t_0, \mu_0, u, v) = \int_{IR^N} g\left(X_T^{t_0,x,u,v}\right) d\mu(x) .$$

For any pair of strategies $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$**, we define**

$$J(t_0, \mu_0, \alpha, \beta) = J(t_0, \mu_0, u_0, v_0)$$

where $(u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ **is associated to** (α, β) **by the Lemma**

Definition of the value functions:

$$V^+(t_0,\mu_0) = \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{\beta\in\mathcal{B}(t_0)} J(t_0,\mu_0,\alpha,\beta)$$

and

$$V^-(t_0,\mu_0) = \sup_{\beta\in\mathcal{B}(t_0)} \inf_{\alpha\in\mathcal{A}(t_0)} J(t_0,\mu_0,\alpha,\beta)\ .$$

Obviously we have

$$V^-(t_0,\mu_0) \le V^+(t_0,\mu_0) \qquad \forall (t_0,\mu_0)\in[0,T]\times\mathcal{M}\ .$$

Remark

$$V^+(t_0,\mu_0) = \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{v\in\mathcal{V}(t_0)} J(t_0,\mu_0,\alpha(v),v)$$

$$V^-(t_0,\mu_0) = \sup_{\beta\in\mathcal{B}(t_0)} \inf_{u\in\mathcal{U}(t_0)} J(t_0,\mu_0,u,\beta(u))\ .$$

Preliminaries on Probability Measures

For $\mu \in \mathcal{M}$, we denote by $L^2_\mu(I\!R^N, I\!R)$ (resp. $L^2_\mu(I\!R^N, I\!R^N)$) the set of $\mu-$measurables maps $p : I\!R^N \to I\!R$ (resp. $p : I\!R^N \to I\!R^N$) such that $\|p\|_{L^2_\mu} := \int_{I\!R^N} |p|^2 d\mu < +\infty$

For $\mu \in \mathcal{M}$ and $\varphi : I\!R^N \to I\!R^N$ a Borel measurable with linear growth, $\varphi\sharp\mu$ is the push-forward of μ by φ,

$$\varphi\sharp\mu(A) = \mu\left(\varphi^{-1}(A)\right) \qquad \forall A \subset I\!R^N, \text{ Borel measurable}$$

or, equivalently, such that, $\forall f : I\!R^N \to I\!R$, Borel measurable and bounded,

$$\int_{I\!R^N} f d(\varphi\sharp\mu) = \int_{I\!R^N} f(\varphi(x)) d\mu(x) \ .$$

Wasserstein Distance cf book of Villani

$$\mathbf{d}(\mu,\nu) = \inf \left\{ \left(\int_{I\!R^{2N}} |x-y|^2 d\gamma \right)^{\frac{1}{2}} \right\} \tag{3}$$

where the infimum is taken over all the probability measures γ in $I\!R^{2N}$ such that

$$\pi_1 \sharp \gamma = \mu \quad \text{and} \quad \pi_2 \sharp \gamma = \nu \ , \tag{4}$$

π_1 and π_2 being respectively the projections on the first and the second coordinates: $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$. A measure γ satisfying (4) is an admissible transport plan from μ to ν. The optimal γ are called optimal plans.

Lemma 3 ***If*** $h : IR^N \to IR$ ***is*** $k-$***Lipschitz continuous, then***

$$\left| \int_{IR^N} h(x) d\mu(x) - \int_{IR^N} h(x) d\nu(x) \right| \le k\mathbf{d}(\mu, \nu) \qquad \forall \mu, \nu \in \mathcal{M} \ .$$

Lemma 4 ***Let*** $\mu, \nu \in \mathcal{M}$ ***and*** γ ***be optimal for*** $\mathbf{d}(\mu, \nu)$***. Then there exist*** $p \in L^2_\mu(IR^N, IR^N)$ ***and*** $q \in L^2_\nu(IR^N, IR^N)$ ***s. t.***

$$\int_{IR^N} < \varphi(x), p(x) > d\mu(x) = \int_{IR^{2N}} < \varphi(x), x - y > d\gamma(x) \qquad (5)$$

$$\int_{IR^N} < \varphi(x), q(x) > d\nu(x) = \int_{IR^{2N}} < \varphi(y), x - y > d\gamma(x) \qquad (6)$$

for any Borel measurable map $\varphi : IR^N \to IR^N$ ***with at most a linear growth.***

<u>proof</u> Let γ be an optimal plan from μ to ν. Then

$$\begin{aligned}\int_{IR^N} h(x)d\mu(x) &= \int_{IR^{2N}} h(x)d\gamma(x,y) \\ &\le \int_{IR^{2N}} h(y)d\gamma(x,y) + k\int_{IR^{2N}} |x-y|d\gamma(x,y) \\ &\le \int_{IR^{2N}} h(y)d\nu(y) + k\mathbf{d}(\mu,\nu)\end{aligned}$$

$\square$

<u>proof</u> We just show the existence of p, since the proof for q can be obtained in the same way. Let us consider the linear map Φ on $L^2_\mu(IR^N, IR^N)$ defined by

$$\Phi(\varphi) = \int_{IR^{2N}} < \varphi(x), x-y > d\gamma(x)$$

Then

$$|\Phi(\varphi)| \le \left(\int_{IR^{2N}} |\varphi(x)|^2 d\gamma(x)\right)^{\frac{1}{2}} \left(\int_{IR^{2N}} |x-y|^2 d\gamma(x)\right)^{\frac{1}{2}} \le \mathbf{d}(\mu,\nu)\|\varphi\|_{L^2_\mu}$$

for any $\varphi \in L^2_\mu(IR^N, IR^N)$. Therefore Φ is bounded on $L^2_\mu(IR^N, IR^N)$ whence the existence of $p \in L^2_\mu(IR^N, IR^N)$ from Riesz Representation Theorem. $\square$

Regularity of the Values

Proposition 5 *The value functions* V^+ *and* V^- *are Lipschitz continuous.*

proof for V^+

We shall first prove that the values are Lipschitz continuous with respect to the second variable. Fix $t_0 \in [0,T]$, $\mu_0 \in \mathcal{M}$, $\nu_0 \in \mathcal{M}$ and $\varepsilon > 0$. There exists an nonanticipative strategy $\alpha_\varepsilon \in \mathcal{A}(t_0)$ such that

$$V^+(t_0,\nu_0) \leq \sup_{\beta \in \mathcal{B}(t_0)} J(t_0,\nu_0,\alpha_\varepsilon,\beta) \leq V^+(t_0,\nu_0) + \varepsilon.$$

Hence

$$V^+(t_0,\mu_0) - V^+(t_0,\nu_0) \leq$$

$$\varepsilon + \sup_{\beta \in \mathcal{B}(t_0)} J(t_0,\mu_0,\alpha_\varepsilon,\beta) - \sup_{\beta \in \mathcal{B}(t_0)} J(t_0,\nu_0,\alpha_\varepsilon,\beta)$$

$$\leq 2\varepsilon + J(t_0,\mu_0,\alpha_\varepsilon,\beta_\varepsilon) - J(t_0,\nu_0,\alpha_\varepsilon,\beta_\varepsilon)$$

where $\beta_\varepsilon \in \mathcal{B}(t_0)$ **is such that**

$$\sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta) - \varepsilon \leq J(t_0, \mu_0, \alpha_\varepsilon, \beta_\varepsilon) \leq \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta).$$

Thus

$$V^+(t_0, \mu_0) - V^+(t_0, \nu_0) \leq$$

$$2\varepsilon + \int_{IR^N} g\left(X_T^{t_0,x,\alpha_\varepsilon,\beta_\varepsilon}\right) d\mu_0(x) - \int_{IR^N} g\left(X_T^{t_0,x,\alpha_\varepsilon,\beta_\varepsilon}\right) d\nu_0(x)$$

$$\leq k^2 e^{kT} d(\mu_0, \nu_0) + 2\varepsilon$$

thanks to Lemma 3 and because $x \mapsto X_T^{t_0,x,u,v}$ **is** $k^2 e^{kT}$ **Lipschitz.**

Consider now $0 < t_0 < s_0 < T$ **and the strategy** α_ε **. Let** $u_0 \in \mathcal{U}(t_0)$ **and** $v_0 \in \mathcal{V}(t_0)$ **two given control. We define the**

nonanticipative strategy $\alpha_1 \in A(s_0)$ **as follows**

$$\forall v \in \mathcal{V}(s_0),\ \alpha_1(v) := \alpha_\varepsilon(v_1)$$

where $v_1(t) = v_0(t)$ **if** $t \in [t_0, s_0)$ **and** $v_1(t) = v(t)$ **if** $t \in [s_0, T]$**.**

$$\begin{aligned}
&V^+(s_0,\mu_0) - V^+(t_0,\mu_0)\\
&\leq \varepsilon + \sup_{\beta\in\mathcal{B}(s_0)} J(s_0,\mu_0,\alpha_1,\beta) - \sup_{\beta\in\mathcal{B}(t_0)} J(t_0,\mu_0,\alpha_\varepsilon,\beta)\\
&\leq 2\varepsilon + J(s_0,\mu_0,\alpha_1,\beta_\varepsilon) - J(t_0,\mu_0,\alpha_\varepsilon,\beta_1)
\end{aligned}$$

where $\beta_\varepsilon \in \mathcal{B}(s_0)$ **is such that**

$$\sup_{\beta\in\mathcal{B}(s_0)} J(s_0,\mu_0,\alpha_1,\beta) - \varepsilon \leq J(s_0,\mu_0,\alpha_1,\beta_\varepsilon) \leq \sup_{\beta\in\mathcal{B}(s_0)} J(s_0,\mu_0,\alpha_1,\beta)$$

and $\beta_1 \in B(t_0)$ **is defined as follows:**

$\forall u \in \mathcal{U}(t_0),\ \ \beta_1(u)(t) = v_0(t)$ **if** $t \in [t_0, s_0)$ **and** $\beta_1(u)|_{[s_0,T]} = \beta_\varepsilon(u|_{[s_0,T]})$**. Hence**

$$
\begin{aligned}
&V^+(s_0,\mu_0) - V^+(t_0,\mu_0) \\
&\leq 2\varepsilon + \int_{I\!R^N} g\left(X_T^{s_0,x,\alpha_1,\beta_\varepsilon}\right) d\mu_0(x) - \int_{I\!R^N} g\left(X_T^{t_0,x,\alpha_\varepsilon,\beta_1}\right) d\mu_0(x) \\
&= 2\varepsilon + \int_{I\!R^N} g\left(X_T^{s_0,x,\alpha_1,\beta_\varepsilon}\right) - g\left(X_T^{s_0,X_{s_0}^{t_0,x,\alpha_1,v_0},\alpha_1,\beta_\varepsilon}\right) d\mu_0(x)
\end{aligned}
$$

by noticing that

$$
X_T^{t_0,x,\alpha_\varepsilon,\beta_1} = X_T^{s_0,X_{s_0}^{t_0,x,\alpha_1,v_0},\alpha_1,\beta_\varepsilon}.
$$

Consequently

$$
V^+(s_0,\mu_0) - V^+(t_0,\mu_0) \leq 2\varepsilon + Mk^2e^{kT}(t_0 - s_0),
$$

using the fact that $|x - X_{s_0}^{t_0,x,\alpha_1,v_0}| \leq M(t_0 - s_0)|$ $\square$

Proposition 6 *[Dynamic programming] Let* $(t_0, t_1, \mu_0) \in [0, T) \times [0, T] \times \mathcal{M}$ ***be fixed with*** $t_0 < t_1$***. Then***

$$V^+(t_0, \mu_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} V^+ \left(t_1, X_{t_1}^{t_0, \cdot, \alpha, \beta} \sharp \mu_0 \right)$$

and

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} V^- \left(t_1, X_{t_1}^{t_0, \cdot, \alpha, \beta} \sharp \mu_0 \right)$$

proof We only prove the dynamic programming for

$$V^-(t_0,\mu_0) = \sup_{\beta\in\mathcal{B}(t_0)} \inf_{u\in\mathcal{U}(t_0)} J(t_0,\mu_0,u,\beta(u)) \ .$$

$V^-(t_0,\mu_0) = W(t_0,t_1,\mu_0)$ **where we set**

$$W(t_0,t_1,\mu_0) = \sup_{\beta\in\mathcal{B}(t_0)} \inf_{u\in\mathcal{U}(t_0)} V^-\left(t_1, X_{t_1}^{t_0,\cdot,u,\beta}\sharp\mu_0\right)$$

Let us prove first that $V^-(t_0,\mu_0) \le W(t_0,t_1,\mu_0)$**. Fix** $\beta_0 \in \mathcal{B}(t_0)$ **and** $u_0 \in \mathcal{U}(t_0)$**. Define** $\beta_1 \in \mathcal{B}(t_1)$ **as follows**

$$\forall u \in \mathcal{U}(t_1),\ \beta_1(u) := \beta_0(u_1)$$

where $u_1(t) = u_0(t)$ **if** $t \in [t_0,t_1)$ **and** $u_1(t) = u(t)$ **if** $t \in [s_0,T]$**.**

Clearly β_1 is nonanticipative such that

$$\forall t \in [t_1, T],\ X_t^{t_0,x,u,\beta_0(u)} = X_t^{t_1, X_{t_1}^{t_0,x,u_0,\beta_0(u_0)}, u, \beta_1(u)}.$$

Hence for any $u \in \mathcal{U}(t_1)$,

$$J(t_1, X_{t_1}^{t_0,\cdot,u_0,\beta_0(u_0)} \sharp \mu_0, u, \beta_1) =$$

$$= \int_{I\!R^N} g\left(X_T^{t_1,x,u,\beta_1}\right) d(X_{t_1}^{t_0,\cdot,u_0,\beta_0(u_0)} \sharp \mu_0)(x) =$$

$$\int_{I\!R^N} g\left(X_T^{t_0,x,u_1,\beta_0(u_1)}\right) d\mu_0(x)$$

So

$$\inf_{u \in \mathcal{U}(t_1)} J(t_1, X_{t_1}^{t_0,\cdot,u_0,\beta_0(u_0)} \sharp \mu_0, u, \beta_1) =$$

v

$$\inf_{u \in \mathcal{U}(t_0) \,\textbf{with}\, u|_{[t_0,t_1]} = u_0|_{[t_0,t_1]}} J(t_0, \mu_0, u, \beta_0).$$

Hence

$$V^-(t_1, X_{t_1}^{t_0,\cdot,u_0,\beta_0(u_0)}) \geq \inf_{u \in \mathcal{U}(t_0) \,\textbf{with}\, u|_{[t_0,t_1]} = u_0|_{[t_0,t_1]}} J(t_0, \mu_0, u, \beta_0).$$

Consequently, u_0 **and** β_0 **being arbitrary, we have** $V^-(t_0, \mu_0) \leq W(t_0, t_1, \mu_0)$**.**

Let us prove the reverse inequality

$$V^-(t_0,\mu_0) \geq W(t_0,t_1,\mu_0) := \sup_{\beta\in\mathcal{B}(t_0)} \inf_{u\in\mathcal{U}(t_0)} V^-\left(t_1, X_{t_1}^{t_0,\cdot,u,\beta}\sharp\mu_0\right)$$

Fix $\varepsilon > 0$**. For any** $\mu \in \mathcal{M}$ **there exists some** $\beta_\mu \in \mathcal{B}(t_1)$ **such that**

$$\inf_{u\in\mathcal{U}(t_1)} J(t_1,\mu,u,\beta_\mu) \geq V^-(t_1,\mu) - \varepsilon.$$

Fix $\beta_0 \in \mathcal{U}(t_0)$**. Define** $\beta_0 \in \mathcal{B}(t_0)$ **as follows: for any** $u \in \mathcal{U}(t_0)$ **we have**

$$\beta(u)|_{[t_0,t_1]} = \beta_0(u)|_{[t_0,t_1]}, \quad \beta(u)|_{[t_1,T]} = \beta_{\mu_1}(u|_{[t_1,T]}),$$

where $\mu_1 = X_{t_1}^{t_0,\cdot,u,\beta_0}\sharp\mu_0$**. Hence for any** $u \in \mathcal{U}(t_0)$ **we obtain**

$$J(t_0,\mu_0,u,,\beta(u)) = J(t_1,\mu_1,u|_{[t_1,T]},\beta_{\mu_1}(u|_{[t_1,T]})) \geq V^-(t_1,\mu_1) - \varepsilon.$$

Hence

$$V^{-}(t_0,\mu_0) \geq \inf_{u\in\mathcal{U}(t_0)} J(t_0,\mu_0,u,,\beta(u)) \geq \inf_{u\in\mathcal{U}(t_0)} V^{-}(t_1, X_{t_1}^{t_0,\cdot,u,\beta_0}\sharp\mu_0) - \varepsilon.$$

We obtained the wished conclusion passing to the supremum in β_0 because ε is arbitrary. □

Hamilton Jacobi Isaacs Equation

$$w_t + \mathcal{H}(\mu, Dw) = 0 \tag{7}$$

where $\mathcal{H} = \mathcal{H}(\mu, p)$ **is an Hamiltonian defined for any** $\mu \in \mathcal{M}$ **and** $p \in L^2_\mu(I\!R^N, I\!R^N)$**.**

Definition 7 (Sub- and super-differential) ***Let*** $w : [0,T]\times\mathcal{M} \to I\!R$ ***be a function,*** $(t_0,\mu_0) \in (0,T)\times\mathcal{M}$ ***and let*** $\delta > 0$***.*** $(p_t,p_\mu) \in I\!R\times L^2_\mu(I\!R^N, I\!R^N)$ ***belongs to the*** δ***-super-differential*** $D^+_\delta w(t_0,\mu_0)$ ***to*** w ***at*** (t_0,μ_0) ***if,*** $\forall\varphi \in \mathcal{C}_b(I\!R^N, I\!R^N)$***,***

$$\limsup_{\|\varphi\|_{L^2_\mu}\to 0,\, t\to t_0} [w(t,(\mathrm{id}_{\mathbb{R}^N}+\varphi)\sharp\mu_0) - w(t_0,\mu_0) - p_t(t-t_0)$$

$$-\int_{I\!R^N} <\varphi(x), p_\mu(x)> d\mu_0(x)]\frac{1}{\|\varphi\|_{L^2_\mu}+|t-t_0|} \le \delta$$

A pair $(p_t,p_\mu) \in I\!R\times L^2_\mu(I\!R^N, I\!R^N)$ ***belongs to the*** δ***-sub-differential*** $D^-_\delta w(t_0,\mu_0)$ ***to*** w ***at*** (t_0,μ_0) ***if*** $(-p_t,-p_\mu)$ ***belongs to the*** δ***-super-differential to*** $-w$ ***at*** (t_0,μ_0)***.***

Solutions of Hamilton-Jacobi equation

Definition 8 ***We say that a map*** $w : [0,T] \times \mathcal{M} \to I\!R$ ***is a sub-solution of the HJ equation (7) if*** w ***is upper semi-continuous and if, for any*** $(t_0, \mu_0) \in (0,T) \times \mathcal{M}$***, for any*** $(p_t, p_\mu) \in D^+_\delta w(t_0, \mu_0)$***, we have for any*** $\delta > 0$,

$$p_t + \mathcal{H}(\mu_0, p_\mu) \ \geq \ -C\delta \tag{8}$$

where $C > 0$ ***is a constant which depends only of*** $\mathcal{H}$***.***

In a similar way, w ***is a super-solution of the HJ equation (7) if*** w ***is lower semicontinuous and if, for any*** $(t_0, \mu_0) \in (0,T) \times \mathcal{M}$***, for any*** $(p_t, p_\mu) \in D^-_\delta w(t_0, \mu_0)$***, we have***

$$p_t + \mathcal{H}(\mu_0, p_\mu) \ \leq \ C\delta \ . \tag{9}$$

Values and HJI Equations

$$\mathcal{H}^+(\mu, p) = \inf_{u \in U} \sup_{v \in V} \int_{IR^N} < f(x, u, v), p(x) > d\mu(x)$$

$$\mathcal{H}^-(\mu, p) = \sup_{v \in V} \inf_{u \in U} \int_{IR^N} < f(x, u, v), p(x) > d\mu(x)\ .$$

Lemma 9 ***Let*** $\mu, \nu \in \mathcal{M}$***,*** γ ***be an optimal plan from*** μ ***to*** ν***, and*** $p \in L^2_\mu$ ***and*** $q \in L^2_\nu$ ***be defined by (5) and (6) respectively. Then, for*** $\mathcal{H} = \mathcal{H}^+$ ***or*** $\mathcal{H} = \mathcal{H}^-$

$$|\mathcal{H}(\mu, p) - \mathcal{H}(\nu, q)| \leq k(\mathbf{d}(\mu, \nu))^2\ ,$$

$$\forall \varphi,\ \int_{IR^N} < \varphi(x), p(x) > d\mu(x) = \int_{IR^{2N}} < \varphi(x), x - y > d\gamma(x)$$

proof for $\mathcal{H}^-$

$$\begin{aligned}
\mathcal{H}^-(\mu, p) &= \sup_v \inf_u \int_{IR^N} < f(x,u,v), p(x) > d\mu(x) \\
&= \sup_v \inf_u \int_{IR^{2N}} < f(x,u,v), x - y > d\gamma(x,y) \\
&\leq \sup_v \inf_u \int_{IR^{2N}} < f(y,u,v), x - y > d\gamma(x,y) \\
&\qquad\qquad +k \int_{IR^{2N}} |x-y|^2 d\gamma(x,y) \\
&\leq \sup_v \inf_u \int_{IR^N} < f(y,u,v), q(y) > d\nu(y) \\
&\qquad\qquad +k\mathbf{d}^2(\mu,\nu) \\
&\leq \mathcal{H}^-(\nu, q) + k\mathbf{d}^2(\mu,\nu)
\end{aligned}$$

$\square$

Proposition 10 ***The upper value function V^+ is a solution to HJI with $\mathcal{H} := \mathcal{H}^+$ while the lower value function V^- is a solution to HJI with $\mathcal{H} := \mathcal{H}^-$.***

proof of V^+ is a subsolution

Fix $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$, $\delta > 0$ and $(p_t, p_\mu) \in D^+_\delta V^+(t_0, \mu_0)$. We will prove that

$$p_t + \mathcal{H}(\mu_0, p_\mu) \;\geq\; -\delta \tag{10}$$

Consider $t \in (t_0, T)$. For any $\alpha \in \mathcal{A}(t_0)$ and $\beta \in \mathcal{B}(t_0)$ define $\varphi_{\alpha,\beta} \in \mathcal{C}_b(IR^N, IR^N)$ such that

$$(\mathrm{id}_{\mathbb{R}^{\mathrm{N}}} + \varphi_{\alpha,\beta})(x) = X_t^{t_0,x,\alpha,\beta} = x + \int_{t_0}^{t} f(x(s), u(s), v(s))ds,$$

where (u, v) is associated with (α, β) and $x(s) = X_s^{t_0,x,\alpha,\beta}$.

$$V^+(t, X_t^{t_0,\cdot,\alpha,v}\sharp\mu_0) \ - V^+(t_0,\mu_0) - p_t(t-t_0)$$
$$- \int_{IR^N} < \int_{t_0}^t f(X_s^{t_0,x,\alpha,\beta}, u(s), v(s))ds, p_\mu(x) > d\mu(x) \qquad (11)$$
$$\leq (\|\varphi_{\alpha,\beta}\|_{L^2_\mu} + |t-t_0|)(\varepsilon(t,\varphi_{\alpha,\beta}) + \delta)$$

where $\varepsilon(t,\varphi_{\alpha,\beta}) \to 0$ **as** $t \to t_0$ **and** $\varphi_{\alpha,\beta} \to 0$ **in** L^2_μ**. Passing to the sup on** v **and inf on** α**, we obtain by DDP**

$$0 \leq \inf_{\alpha\in\mathcal{A}(t_0)} \sup_{v\in\mathcal{V}(t_0)} [\int_{IR^N} < \int_{t_0}^t f(X_s^{t_0,x,\alpha(v),v}, \alpha(v)(s), v(s))ds, p_\mu(x) > d\mu$$
$$+p_t(t-t_0) + (\|\varphi_{\alpha(v),v}\|_{L^2_\mu} + |t-t_0|)(\delta + \varepsilon(t,\varphi_{\alpha,\beta}))]$$

for $\|\varphi_{\alpha(v),v}\|_{L^2_\mu} + |t-t_0|$ **small enough.**

For t close enough to t_0 we obtain

$$0 \leq \inf_{u \in U} \sup_{v \in \mathcal{V}(t_0)} [\int_{IR^N} < \int_{t_0}^{t} f(x,u,v(s))ds, p_\mu(x) > d\mu(x)$$
$$+p_t(t-t_0) + (\|\varphi_{u,v}\|_{L^2_\mu} + |t-t_0|)(\delta + \varepsilon(t,\varphi_{u,v}))]$$

when we restrict the infimum to nonanticipative strategies α which has constant control values. Hence

$$0 \leq \inf_{u \in U} \sup_{v \in V} [(t-t_0) \int_{IR^N} < f(x,u,v)ds, p_\mu(x) > d\mu(x)$$
$$+p_t(t-t_0) + (\|\varphi_{u,v}\|_{L^2_\mu} + |t-t_0|)(\delta + \varepsilon(t,\varphi_{u,v}))]$$

Dividing this inequality by $t - t_0$ and letting $t \to t_0^+$ gives, since $\|\varphi_{u,v}\|_{L^2_\mu} = O(t-t_0)$,

$$p_t + \mathcal{H}(\mu_0, p_\mu)) \ \geq \ -(1+M)\delta \ .$$

Comparison Principle for HJI

$$w_t + \mathcal{H}(\mu, Dw) = 0$$

Assumptions on $\mathcal{H}$

• $p \in L^2_\mu(I\!R^N, I\!R^N) \mapsto \mathcal{H}(\mu, \cdot)$ is positively homogeneous.

• **for any $\mu, \nu \in \mathcal{M}$, if γ is the optimal plan from μ to ν, and $p \in L^2_\mu$ and $q \in L^2_\nu$ are defined by (5) and (6) respectively,**

$$|\mathcal{H}(\mu, p) - \mathcal{H}(\nu, q)| \leq k(\mathbf{d}(\mu, \nu))^2 \,. \tag{12}$$

$\forall \varphi,\ \int_{I\!R^N} < \varphi(x), p(x) > d\mu(x) = \int_{I\!R^{2N}} < \varphi(x), x - y > d\gamma(x)$
$\forall \varphi,\ \int_{I\!R^N} < \varphi(x), q(x) > d\nu(x) = \int_{I\!R^{2N}} < \varphi(y), x - y > d\gamma(x)$

Comparison principle for HJI

Theorem 11 ***Let*** w_1 ***be a bounded and Lipschitz continuous subsolution and*** w_2 ***be a bounded and Lipschitz continuous supersolution to (7). Then***

$$\inf_{[0,T]\times\mathcal{M}} (w_2 - w_1) = \inf_{\mathcal{M}} \; w_2(T,\cdot) - w_1(T,\cdot) \, .$$

Proof of Comparison Principle

$$A = \inf_{\mu \in \mathcal{M}} \; w_2(T, \mu) - w_1(T, \mu) \; .$$

Since $\mathcal{H}$ is independant of w, $w_1 - A$ is still a subsolution. So we suppose without loss of generality that $A = 0$.

By Contradiction

$$-\xi := \inf_{\mu \in \mathcal{M},\, t \in [0,T]} w_2(t, \mu) - w_1(t, \mu) \; < 0 \; .$$

And choose $(t_0, \mu_0) \in [0, T] \times \mathcal{M}$ such that

$$(w_2 - w_1)(t_0, \mu_0) < -\xi/2. \tag{13}$$

Let $C > 0$ such that $\forall \delta > 0$

$\forall (p_t, p_\mu) \in D_\delta^+ w_1(t_0, \mu_0),\ p_t + \mathcal{H}(\mu_0, p_\mu) \ \geq\ -C\delta$

$\forall (p_t, p_\mu) \in D_\delta^- w_2(t_0, \mu_0)\ p_t + \mathcal{H}(\mu_0, p_\mu) \ \leq\ C\delta\ .$

Fix$\varepsilon > 0$, $\eta > 0$ and $\delta > 0$ sufficiently small such that

$$\xi > 2\eta T + \frac{k^2\varepsilon}{2} \qquad \text{and} \qquad 2C\delta + 2k(\delta + k)^2\varepsilon < \eta\ . \tag{14}$$

We consider the following continuous function defined on $([0, T] \times \mathcal{M})^2$:

$$\Phi(s, \mu, t, \nu) = -w_1(s, \mu) + w_2(t, \nu) + \tfrac{1}{\varepsilon}\left(\mathbf{d}^2(\mu, \nu) + (t - s)^2\right) - \eta s\ .$$

Define

$$(\bar{\mu}, \bar{\nu}, \bar{s}, \bar{t}) \in Arg \min_{[0,T]\times\mathcal{M}} \Phi$$

From Ekeland Variational Principle that $\exists(\bar{\mu}, \bar{\nu}, \bar{s}, \bar{t}) \in \mathcal{M}^2 \times [0,T]^2$ **such that for any** $(s, \mu, t, \nu) \in ([0,T] \times \mathcal{M})^2$

$$\begin{cases} i) \ \ \Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(t_0, \mu_0, t_0, \mu_0) \\ ii) \ \Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(s, \mu, t, \nu) \\ \qquad +\delta([\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} + [\mathbf{d}^2(\nu, \bar{\nu}) + |t - \bar{t}|^2]^{\frac{1}{2}}) \end{cases} \tag{15}$$

__**CLAIM 1**__ $\rho^2 := \mathbf{d}^2(\bar{\mu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2 \leq (k+\delta)^2\varepsilon^2$ **Indeed,**

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(\bar{s}, \bar{\mu}, \bar{s}, \bar{\mu}) + \delta[\mathbf{d}^2(\bar{\nu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2]^{\frac{1}{2}}$$

Since w_2 **is** $k-$**Lipschitz continuous,**

$$\begin{aligned} \delta\rho + w_2(\bar{s}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - \eta\bar{s} &\geq w_2(\bar{t}, \bar{\nu}) - w_1(\bar{s}, \bar{\mu}) + \tfrac{1}{\varepsilon}\rho^2 - \eta\bar{s} \\ &\geq w_2(\bar{s}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - k\rho + \tfrac{1}{\varepsilon}\rho^2 - \eta\bar{s} \end{aligned}$$

Hence $\rho \leq (k+\delta)\varepsilon$**,**

Assume $\bar{s}, \bar{t} \in (0,T)$. Let γ be the optimal transport plan between $\bar{\mu}$ and $\bar{\nu}$ and $p \in L^2(\bar{\mu})$ and $q \in L^2(\bar{\nu})$ associated.

<u>CLAIM 2</u>

$$\left(\frac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta, \frac{2}{\varepsilon}p\right) \in D^+_\delta w_1(\bar{s},\bar{\mu}), \; \left(\frac{2}{\varepsilon}(\bar{s}-\bar{t}), \frac{2}{\varepsilon}q\right) \in D^-_\delta w_2(\bar{t},\bar{\nu}),$$

From (15)-ii), we have for any (s,μ),

$$\Phi(\bar{s},\bar{\mu},\bar{t},\bar{\nu}) \leq \Phi(s,\mu,\bar{t},\bar{\nu}) + \delta[\mathbf{d}^2(\mu,\bar{\mu}) + |s-\bar{s}|^2]^{\frac{1}{2}}.$$

$$\begin{aligned} w_1(s,\mu) \leq\; & w_1(\bar{s},\bar{\mu}) + \tfrac{1}{\varepsilon}(\mathbf{d}^2(\mu,\bar{\nu}) - \mathbf{d}^2(\bar{\mu},\bar{\nu}) + (s-\bar{t})^2 - (\bar{s}-\bar{t})^2) \\ & + \delta[\mathbf{d}^2(\mu,\bar{\mu}) + |s-\bar{s}|^2]^{\frac{1}{2}} + \eta(\bar{s}-s) \end{aligned} \tag{16}$$

We will replace μ by $(\mathrm{id}_{\mathbb{R}^N} + \varphi)\sharp\bar{\mu}$ with $\varphi \in \mathcal{C}_b(IR^N, IR^N)$

Estimation

Observe $\gamma' = (\mathrm{id}_{\mathbb{R}^{\mathrm{N}}} + \varphi, \mathrm{id}_{\mathbb{R}^{\mathrm{N}}})\sharp\gamma$ **is an admissible transport plan from** $(\mathrm{id}_{\mathbb{R}^{\mathrm{N}}} + \varphi)\sharp\bar{\mu}$ **to** $\bar{\nu}$**. Hence**

$$\begin{aligned}
&\mathbf{d}^2((\mathrm{id}_{\mathbb{R}^{\mathrm{N}}} + \varphi)\sharp\bar{\mu}, \bar{\nu}) - \mathbf{d}^2(\bar{\mu}, \bar{\nu}) \ \leq \\
&\int_{IR^{2N}} |x-y|^2 d\gamma'(x,y) - \int_{IR^{2N}} |x-y|^2 d\gamma(x,y) \\
&= \int_{IR^{2N}} (|x+\varphi(x)-y|^2 - |x-y|^2) d\gamma(x,y) \\
&= \int_{IR^{2N}} (2 < x-y, \varphi(x) > + |\varphi(x)|^2) d\gamma(x,y) \\
&= \int_{IR^N} < 2p(x), \varphi(x) > d\bar{\mu}(x) + \|\varphi\|^2_{L^2_{\bar{\mu}}} \ \textbf{(from the def. of } p\textbf{)}
\end{aligned}$$

$$w_1(s, (\mathrm{id}_{\mathbb{R}^{\mathrm{N}}} + \varphi)\sharp\bar{\mu}) \leq w_1(\bar{s}, \bar{\mu}) + \frac{1}{\varepsilon}\int_{IR^N} < 2p(x), \varphi(x) > d\bar{\mu}(x) + \|\varphi\|^2_{L^2_{\bar{\mu}}}$$

$$+\frac{1}{\varepsilon}(s-\bar{s})[(s-\bar{s}) + 2(\bar{s}-\bar{t})] + \delta[\mathbf{d}^2(\mu,\bar{\mu}) + |s-\bar{s}|^2]^{\frac{1}{2}} + \eta(\bar{s}-s)$$

So

$$w_1(s,(\mathrm{id}_{\mathbb{R}^{\mathrm{N}}}+\varphi)\sharp\bar{\mu})-w_1(\bar{s},\bar{\mu})-(s-\bar{s})[\tfrac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta]$$
$$-\tfrac{1}{\varepsilon}\int_{I\!R^N}<2p(x),\varphi(x)>d\bar{\mu}(x)+\|\varphi\|^2_{L^2_{\bar{\mu}}}\leq)$$
$$+\tfrac{1}{\varepsilon}(s-\bar{s})^2+\delta[\mathbf{d}^2(\mu,\bar{\mu})+|s-\bar{s}|^2]^{\frac{1}{2}}$$

Hence

$$\left(\frac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta,\frac{2}{\varepsilon}p\right)\in D^+_\delta w_1(\bar{s},\bar{\mu})$$

which the claim 2.

By Claim 2, because w_1 **sub solution,** w_2 **supersolution**

$$\frac{2}{\varepsilon}(\bar{s}-\bar{t})-\eta+\mathcal{H}\left(\bar{\mu},\frac{2}{\varepsilon}p\right)\geq -C\delta\ .$$

$$\frac{2}{\varepsilon}(\bar{s}-\bar{t})+\mathcal{H}\left(\bar{\mu},\frac{2}{\varepsilon}q\right)\leq C\delta\ .$$

Substracting we obtain $\mathcal{H}(\bar{\nu},q)-\mathcal{H}(\bar{\mu},p)\leq 2C\delta-\eta$.

But from the assumtion on $\mathcal{H}$ **we have** $\mathcal{H}(\bar{\nu},q)-\mathcal{H}(\bar{\mu},p)\geq -k\mathbf{d}^2(\bar{\mu},\bar{\nu})$. **So**

$$-k(k+\delta)^2\varepsilon\leq 2C\delta-\eta\ ,$$

a contradiction with (14).

We have to check that $\bar{s}$ and $\bar{t}$ cannot be equal to 0 or T. Let us assume for instance that $\bar{s} = T$. We first note that

$$\Phi(t_0, \mu_0, t_0, \mu_0) = w_2(t_0, \mu_0) - w_1(t_0, \mu_0) - \eta t_0 \leq -\xi/2$$

From (Ekeland), i), we have

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(t_0, \mu_0, t_0, \mu_0) \leq -\xi/2 \ .$$

Since $\bar{s} = T$ and w_2 is $k-$Lipschitz continuous, we get (setting as before $\rho := [\mathbf{d}^2(\bar{\nu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2]^{\frac{1}{2}}$)

$$\begin{aligned} -\xi/2 \ &\geq \ w_2(\bar{t}, \bar{\nu}) - w_1(T, \bar{\mu}) + \tfrac{1}{\varepsilon}\rho^2 - \eta T \\ &\geq \ w_2(T, \bar{\mu}) - w_1(T, \bar{\mu}) - k\rho + \tfrac{1}{\varepsilon}\rho^2 - \eta T \\ &\geq \ -k\rho + \tfrac{1}{\varepsilon}\rho^2 - \eta T \ , \end{aligned}$$

A contradiction with the choices of $\eta, \delta, \varepsilon$ and the estimation of ρ.

To show that $\bar{s} \neq 0$ and $\bar{t} \neq 0$, it is enough to use the standard fact that w_1 and w_2 are respectively sub- and supersolutions up to $t = 0$,

Lemma 12 ***If*** w ***is a subsolution (resp. a supersolution) of (7) on the time interval*** $(0, T)$***, then*** w ***is also a subsolution (resp. a supersolution) on*** $[0, T)$***.***

□

Corollary 13 ***There exists at most one lipschitz continuous solution of (7) satisfying***

$$w(T, \mu) = \int_{IR^N} g(x) d\mu(x) \ , \forall \mu \in \mathcal{M}.$$

Existence of A value

Theorem 14 ***We suppose the following Isaacs condition:***

$$\mathcal{H}^+ = \mathcal{H}^-.$$

Then the game has a value. Namely:

$$V^+(t,\mu) = V^-(t,\mu) \qquad \forall (t,\mu) \in [0,T] \times \mathcal{M} \; .$$

Furthermore $V^+ = V^-$ ***is the unique solution of the Hamilton-Jacobi equation (7) with*** $\mathcal{H} = \mathcal{H}^+ = \mathcal{H}^-$***.***

Thank You for your Attention