



# Tutorial on Differential Games

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# Differential Games III

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Joint Work P. Cardaliaguet , M.Q

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## The differential Game

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$$\begin{cases} x'(t) = f(x(t), u(t), v(t)), & t \in [0, T] \\ u(t) \in U, v(t) \in V \end{cases} \quad (1)$$

where  $f : \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}^N$ ,  $U$  and  $V$  being the control sets of the players. To any initial condition  $x(t_0) = x_0$  we associate  $t \rightarrow X_t^{t_0, x_0, u, v}$  the solution to (1).

The first player—choosing  $u$ —wants to minimize a final cost of the form

$$g(x(T))$$

while the second player,—playing with  $v$ —wants to maximize it.

the state-space  $x_0$  is only imperfectly known by the players : they only know that the initial position is randomly distributed under some fixed probability measure  $\mu_0$ . Both players are assumed to know this probability  $\mu_0$ , and have a perfect knowledge of the control of the other player.

So the "lack" of information is very specific :

- it is symmetric for both player
- it is only concerned with the current position of the game.

We denote by  $\mathcal{M}$  the Borel probability measures  $\mu$  s.t.

$$\int_{\mathbb{R}^N} |x|^2 d\mu(x) < +\infty .$$

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## Assumptions

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$$\left\{ \begin{array}{l} (i) \quad U \text{ and } V \text{ are compact subsets of some finite} \\ \quad \text{dimensional spaces} \\ (ii) \quad f : \mathbb{R}^N \times U \times V \rightarrow \mathbb{R}^N \text{ is continuous and} \\ \quad \text{Lipschitz continuous with respect to} \\ (iii) \quad \forall (x, u, v), |f(x, u, v)| \leq M \\ (iv) \quad g : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is Lipschitz continuous and bounded} \end{array} \right. \quad (2)$$

$$\mathcal{U}(t_0) = \{u : [t_0, T] \rightarrow U, \text{ Lebesgue measurable}\}$$

$$\mathcal{V}(t_0) = \{v : [t_0, T] \rightarrow V, \text{ Lebesgue measurable}\}$$

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## Strategies

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**Definition 1** A *NAD strategy* is a map  $\alpha : \mathcal{V}(t_0) \rightarrow \mathcal{U}(t_0)$  such that there is some  $\tau > 0$  such that  $\forall t$ ,

$\forall v_1, v_2 \in \mathcal{V}(t_0)$ ,  $v_1 = v_2$  on  $[t_0, t] \Rightarrow \alpha(v_1) = \alpha(v_2)$  on  $[t_0, t + \tau]$ .

$\mathcal{A}(t_0)$  is the set of such  $\alpha$ . Symmetrically,  $\mathcal{B}(t_0)$  is the set of *NAD strategies*  $\beta : \mathcal{U}(t_0) \rightarrow \mathcal{V}(t_0)$  for the second player.

**Lemma 2**  $\forall (\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$ ,  $\exists!(u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ ,

$$\alpha(v_0) = u_0 \text{ and } \beta(u_0) = v_0 .$$

$$X_t^{t_0, x, \alpha, \beta} := X_t^{t_0, x, u_0, v_0} \quad \forall t \in [t_0, T] .$$

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## Payoffs and Values

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$g : \mathbb{R}^N \mapsto \mathbb{R}$  which is Lipschitz and bounded. For any  $(t_0, \mu_0) \in [0, T) \times \mathcal{M}$  and for any  $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$  we set

$$J(t_0, \mu_0, u, v) = \int_{\mathbb{R}^N} g \left( X_T^{t_0, x, u, v} \right) d\mu(x) .$$

For any pair of strategies  $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$ , we define

$$J(t_0, \mu_0, \alpha, \beta) = J(t_0, \mu_0, u_0, v_0)$$

where  $(u_0, v_0) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$  is associated to  $(\alpha, \beta)$  by the Lemma



## Definition of the value functions:

$$V^+(t_0, \mu_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha, \beta)$$

and

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} J(t_0, \mu_0, \alpha, \beta) .$$

Obviously we have

$$V^-(t_0, \mu_0) \leq V^+(t_0, \mu_0) \quad \forall (t_0, \mu_0) \in [0, T] \times \mathcal{M} .$$

Remark

$$V^+(t_0, \mu_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} J(t_0, \mu_0, \alpha(v), v)$$

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} J(t_0, \mu_0, u, \beta(u)) .$$

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## Preliminaries on Probability Measures

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For  $\mu \in \mathcal{M}$ , we denote by  $L^2_\mu(\mathbb{R}^N, \mathbb{R})$  (resp.  $L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ ) the set of  $\mu$ -measurable maps  $p : \mathbb{R}^N \rightarrow \mathbb{R}$  (resp.  $p : \mathbb{R}^N \rightarrow \mathbb{R}^N$ ) such that  $\|p\|_{L^2_\mu} := \int_{\mathbb{R}^N} |p|^2 d\mu < +\infty$

For  $\mu \in \mathcal{M}$  and  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}^N$  a Borel measurable with linear growth,  $\varphi\#\mu$  is the push-forward of  $\mu$  by  $\varphi$ ,

$$\varphi\#\mu(A) = \mu\left(\varphi^{-1}(A)\right) \quad \forall A \subset \mathbb{R}^N, \text{ Borel measurable}$$

or, equivalently, such that,  $\forall f : \mathbb{R}^N \rightarrow \mathbb{R}$ , Borel measurable and bounded,

$$\int_{\mathbb{R}^N} f d(\varphi\#\mu) = \int_{\mathbb{R}^N} f(\varphi(x)) d\mu(x) .$$

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## Wasserstein Distance **cf book of Villani**

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$$\mathbf{d}(\mu, \nu) = \inf \left\{ \left( \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma \right)^{\frac{1}{2}} \right\} \quad (3)$$

where the infimum is taken over all the probability measures  $\gamma$  in  $\mathbb{R}^{2N}$  such that

$$\pi_1\#\gamma = \mu \quad \text{and} \quad \pi_2\#\gamma = \nu, \quad (4)$$

$\pi_1$  and  $\pi_2$  being respectively the projections on the first and the second coordinates:  $\pi_1(x, y) = x$  and  $\pi_2(x, y) = y$ . A measure  $\gamma$  satisfying (4) is an admissible transport plan from  $\mu$  to  $\nu$ . The optimal  $\gamma$  are called optimal plans.

**Lemma 3** *If  $h : \mathbb{R}^N \rightarrow \mathbb{R}$  is  $k$ -Lipschitz continuous, then*

$$\left| \int_{\mathbb{R}^N} h(x) d\mu(x) - \int_{\mathbb{R}^N} h(x) d\nu(x) \right| \leq k \mathbf{d}(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{M}.$$

**Lemma 4** *Let  $\mu, \nu \in \mathcal{M}$  and  $\gamma$  be optimal for  $\mathbf{d}(\mu, \nu)$ . Then there exist  $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$  and  $q \in L^2_\nu(\mathbb{R}^N, \mathbb{R}^N)$  s. t.*

$$\int_{\mathbb{R}^N} \langle \varphi(x), p(x) \rangle d\mu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x) \quad (5)$$

$$\int_{\mathbb{R}^N} \langle \varphi(x), q(x) \rangle d\nu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(y), x - y \rangle d\gamma(x) \quad (6)$$

*for any Borel measurable map  $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}^N$  with at most a linear growth.*

proof Let  $\gamma$  be an optimal plan from  $\mu$  to  $\nu$ . Then

$$\begin{aligned}\int_{\mathbb{R}^N} h(x) d\mu(x) &= \int_{\mathbb{R}^{2N}} h(x) d\gamma(x, y) \\ &\leq \int_{\mathbb{R}^{2N}} h(y) d\gamma(x, y) + k \int_{\mathbb{R}^{2N}} |x - y| d\gamma(x, y) \\ &\leq \int_{\mathbb{R}^{2N}} h(y) d\nu(y) + k \mathbf{d}(\mu, \nu)\end{aligned}$$

□

proof We just show the existence of  $p$ , since the proof for  $q$  can be obtained in the same way. Let us consider the linear map  $\Phi$  on  $L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$  defined by

$$\Phi(\varphi) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x)$$

Then

$$|\Phi(\varphi)| \leq \left( \int_{\mathbb{R}^{2N}} |\varphi(x)|^2 d\gamma(x) \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma(x) \right)^{\frac{1}{2}} \leq \mathbf{d}(\mu, \nu) \|\varphi\|_{L^2_\mu}$$

for any  $\varphi \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ . Therefore  $\Phi$  is bounded on  $L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$  whence the existence of  $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$  from Riesz Representation Theorem.  $\square$

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## Regularity of the Values

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**Proposition 5** *The value functions  $V^+$  and  $V^-$  are Lipschitz continuous.*

## proof for $V^+$

We shall first prove that the values are Lipschitz continuous with respect to the second variable. Fix  $t_0 \in [0, T]$ ,  $\mu_0 \in \mathcal{M}$ ,  $\nu_0 \in \mathcal{M}$  and  $\varepsilon > 0$ . There exists a nonanticipative strategy  $\alpha_\varepsilon \in \mathcal{A}(t_0)$  such that

$$V^+(t_0, \nu_0) \leq \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \nu_0, \alpha_\varepsilon, \beta) \leq V^+(t_0, \nu_0) + \varepsilon.$$

Hence

$$\begin{aligned} & V^+(t_0, \mu_0) - V^+(t_0, \nu_0) \leq \\ & \varepsilon + \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta) - \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \nu_0, \alpha_\varepsilon, \beta) \\ & \leq 2\varepsilon + J(t_0, \mu_0, \alpha_\varepsilon, \beta_\varepsilon) - J(t_0, \nu_0, \alpha_\varepsilon, \beta_\varepsilon) \end{aligned}$$



where  $\beta_\varepsilon \in \mathcal{B}(t_0)$  is such that

$$\sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta) - \varepsilon \leq J(t_0, \mu_0, \alpha_\varepsilon, \beta_\varepsilon) \leq \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta).$$

Thus

$$\begin{aligned} V^+(t_0, \mu_0) - V^+(t_0, \nu_0) &\leq \\ 2\varepsilon + \int_{\mathbb{R}^N} g \left( X_T^{t_0, x, \alpha_\varepsilon, \beta_\varepsilon} \right) d\mu_0(x) - \int_{\mathbb{R}^N} g \left( X_T^{t_0, x, \alpha_\varepsilon, \beta_\varepsilon} \right) d\nu_0(x) \\ &\leq k^2 e^{kT} d(\mu_0, \nu_0) + 2\varepsilon \end{aligned}$$

thanks to Lemma 3 and because  $x \mapsto X_T^{t_0, x, u, v}$  is  $k^2 e^{kT}$  Lipschitz.

Consider now  $0 < t_0 < s_0 < T$  and the strategy  $\alpha_\varepsilon$ . Let  $u_0 \in \mathcal{U}(t_0)$  and  $v_0 \in \mathcal{V}(t_0)$  two given control. We define the

**nonanticipative strategy  $\alpha_1 \in A(s_0)$  as follows**

$$\forall v \in \mathcal{V}(s_0), \alpha_1(v) := \alpha_\varepsilon(v_1)$$

**where  $v_1(t) = v_0(t)$  if  $t \in [t_0, s_0)$  and  $v_1(t) = v(t)$  if  $t \in [s_0, T]$ .**

$$\begin{aligned} & V^+(s_0, \mu_0) - V^+(t_0, \mu_0) \\ & \leq \varepsilon + \sup_{\beta \in \mathcal{B}(s_0)} J(s_0, \mu_0, \alpha_1, \beta) - \sup_{\beta \in \mathcal{B}(t_0)} J(t_0, \mu_0, \alpha_\varepsilon, \beta) \\ & \leq 2\varepsilon + J(s_0, \mu_0, \alpha_1, \beta_\varepsilon) - J(t_0, \mu_0, \alpha_\varepsilon, \beta_1) \end{aligned}$$

**where  $\beta_\varepsilon \in \mathcal{B}(s_0)$  is such that**

$$\sup_{\beta \in \mathcal{B}(s_0)} J(s_0, \mu_0, \alpha_1, \beta) - \varepsilon \leq J(s_0, \mu_0, \alpha_1, \beta_\varepsilon) \leq \sup_{\beta \in \mathcal{B}(s_0)} J(s_0, \mu_0, \alpha_1, \beta)$$

**and  $\beta_1 \in \mathcal{B}(t_0)$  is defined as follows:**

$\forall u \in \mathcal{U}(t_0), \beta_1(u)(t) = v_0(t)$  if  $t \in [t_0, s_0)$  and  $\beta_1(u)|_{[s_0, T]} = \beta_\varepsilon(u|_{[s_0, T]})$ . **Hence**

$$\begin{aligned}
& V^+(s_0, \mu_0) - V^+(t_0, \mu_0) \\
& \leq 2\varepsilon + \int_{\mathbb{R}^N} g \left( X_T^{s_0, x, \alpha_1, \beta_\varepsilon} \right) d\mu_0(x) - \int_{\mathbb{R}^N} g \left( X_T^{t_0, x, \alpha_\varepsilon, \beta_1} \right) d\mu_0(x) \\
& = 2\varepsilon + \int_{\mathbb{R}^N} g \left( X_T^{s_0, x, \alpha_1, \beta_\varepsilon} \right) - g \left( X_T^{s_0, X_{s_0}^{t_0, x, \alpha_1, v_0}, \alpha_1, \beta_\varepsilon} \right) d\mu_0(x)
\end{aligned}$$

by noticing that

$$X_T^{t_0, x, \alpha_\varepsilon, \beta_1} = X_T^{s_0, X_{s_0}^{t_0, x, \alpha_1, v_0}, \alpha_1, \beta_\varepsilon}.$$

Consequently

$$V^+(s_0, \mu_0) - V^+(t_0, \mu_0) \leq 2\varepsilon + Mk^2 e^{kT} (t_0 - s_0),$$

using the fact that  $|x - X_{s_0}^{t_0, x, \alpha_1, v_0}| \leq M(t_0 - s_0)$  □

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# Dynamic Programming

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**Proposition 6 [Dynamic programming]** *Let  $(t_0, t_1, \mu_0) \in [0, T) \times [0, T] \times \mathcal{M}$  be fixed with  $t_0 < t_1$ . Then*

$$V^+(t_0, \mu_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} V^+ \left( t_1, X_{t_1}^{t_0, \cdot, \alpha, \beta} \# \mu_0 \right)$$

*and*

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} V^- \left( t_1, X_{t_1}^{t_0, \cdot, \alpha, \beta} \# \mu_0 \right)$$

proof We only prove the dynamic programming for

$$V^-(t_0, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} J(t_0, \mu_0, u, \beta(u)) .$$

$V^-(t_0, \mu_0) = W(t_0, t_1, \mu_0)$  where we set

$$W(t_0, t_1, \mu_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} V^- \left( t_1, X_{t_1}^{t_0, \cdot, u, \beta} \# \mu_0 \right)$$

Let us prove first that  $V^-(t_0, \mu_0) \leq W(t_0, t_1, \mu_0)$ . Fix  $\beta_0 \in \mathcal{B}(t_0)$  and  $u_0 \in \mathcal{U}(t_0)$ . Define  $\beta_1 \in \mathcal{B}(t_1)$  as follows

$$\forall u \in \mathcal{U}(t_1), \beta_1(u) := \beta_0(u_1)$$

where  $u_1(t) = u_0(t)$  if  $t \in [t_0, t_1)$  and  $u_1(t) = u(t)$  if  $t \in [s_0, T]$ .

Clearly  $\beta_1$  is nonanticipative such that

$$\forall t \in [t_1, T], X_t^{t_0, x, u, \beta_0(u)} = X_t^{t_1, X_{t_1}^{t_0, x, u_0, \beta_0(u_0)}, u, \beta_1(u)}.$$

Hence for any  $u \in \mathcal{U}(t_1)$ ,

$$\begin{aligned} & J(t_1, X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)} \# \mu_0, u, \beta_1) = \\ &= \int_{\mathbb{R}^N} g \left( X_T^{t_1, x, u, \beta_1} \right) d(X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)} \# \mu_0)(x) = \\ & \int_{\mathbb{R}^N} g \left( X_T^{t_0, x, u_1, \beta_0(u_1)} \right) d\mu_0(x) \end{aligned}$$

So

$$\inf_{u \in \mathcal{U}(t_1)} J(t_1, X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)} \# \mu_0, u, \beta_1) =$$

**v**

$$\inf_{u \in \mathcal{U}(t_0) \text{ with } u|_{[t_0, t_1]} = u_0|_{[t_0, t_1]}} J(t_0, \mu_0, u, \beta_0).$$

**Hence**

$$V^-(t_1, X_{t_1}^{t_0, \cdot, u_0, \beta_0(u_0)}) \geq \inf_{u \in \mathcal{U}(t_0) \text{ with } u|_{[t_0, t_1]} = u_0|_{[t_0, t_1]}} J(t_0, \mu_0, u, \beta_0).$$

**Consequently,  $u_0$  and  $\beta_0$  being arbitrary, we have  $V^-(t_0, \mu_0) \leq W(t_0, t_1, \mu_0)$ .**

**Let us prove the reverse inequality**

$$V^-(t_0, \mu_0) \geq W(t_0, t_1, \mu_0) := \sup_{\beta \in \mathcal{B}(t_0)} \inf_{u \in \mathcal{U}(t_0)} V^-\left(t_1, X_{t_1}^{t_0, \cdot, u, \beta} \# \mu_0\right)$$

**Fix  $\varepsilon > 0$ . For any  $\mu \in \mathcal{M}$  there exists some  $\beta_\mu \in \mathcal{B}(t_1)$  such that**

$$\inf_{u \in \mathcal{U}(t_1)} J(t_1, \mu, u, \beta_\mu) \geq V^-(t_1, \mu) - \varepsilon.$$

**Fix  $\beta_0 \in \mathcal{U}(t_0)$ . Define  $\beta_0 \in \mathcal{B}(t_0)$  as follows: for any  $u \in \mathcal{U}(t_0)$  we have**

$$\beta(u)|_{[t_0, t_1]} = \beta_0(u)|_{[t_0, t_1]}, \quad \beta(u)|_{[t_1, T]} = \beta_{\mu_1}(u|_{[t_1, T]}),$$

**where  $\mu_1 = X_{t_1}^{t_0, \cdot, u, \beta_0} \# \mu_0$ . Hence for any  $u \in \mathcal{U}(t_0)$  we obtain**

$$J(t_0, \mu_0, u, \beta(u)) = J(t_1, \mu_1, u|_{[t_1, T]}, \beta_{\mu_1}(u|_{[t_1, T]})) \geq V^-(t_1, \mu_1) - \varepsilon.$$



**Hence**

$$V^-(t_0, \mu_0) \geq \inf_{u \in \mathcal{U}(t_0)} J(t_0, \mu_0, u, \beta(u)) \geq \inf_{u \in \mathcal{U}(t_0)} V^-(t_1, X_{t_1}^{t_0, \cdot, u, \beta_0} \# \mu_0) - \varepsilon.$$

**We obtained the wished conclusion passing to the supremum in  $\beta_0$  because  $\varepsilon$  is arbitrary. □**

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# Hamilton Jacobi Isaacs Equation

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$$w_t + \mathcal{H}(\mu, Dw) = 0 \tag{7}$$

where  $\mathcal{H} = \mathcal{H}(\mu, p)$  is an Hamiltonian defined for any  $\mu \in \mathcal{M}$  and  $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$ .

**Definition 7 (Sub- and super-differential)** *Let  $w : [0, T] \times \mathcal{M} \rightarrow \mathbb{R}$  be a function,  $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$  and let  $\delta > 0$ .  $(p_t, p_\mu) \in \mathbb{R} \times L_\mu^2(\mathbb{R}^N, \mathbb{R}^N)$  belongs to the  $\delta$ -super-differential  $D_\delta^+ w(t_0, \mu_0)$  to  $w$  at  $(t_0, \mu_0)$  if,  $\forall \varphi \in \mathcal{C}_b(\mathbb{R}^N, \mathbb{R}^N)$ ,*

$$\limsup_{\|\varphi\|_{L_\mu^2} \rightarrow 0, t \rightarrow t_0} [w(t, (\text{id}_{\mathbb{R}^N} + \varphi) \# \mu_0) - w(t_0, \mu_0) - p_t(t - t_0)$$

$$- \int_{\mathbb{R}^N} \langle \varphi(x), p_\mu(x) \rangle d\mu_0(x)] \frac{1}{\|\varphi\|_{L_\mu^2} + |t - t_0|} \leq \delta$$

*A pair  $(p_t, p_\mu) \in \mathbb{R} \times L_\mu^2(\mathbb{R}^N, \mathbb{R}^N)$  belongs to the  $\delta$ -sub-differential  $D_\delta^- w(t_0, \mu_0)$  to  $w$  at  $(t_0, \mu_0)$  if  $(-p_t, -p_\mu)$  belongs to the  $\delta$ -super-differential to  $-w$  at  $(t_0, \mu_0)$ .*

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## Solutions of Hamilton-Jacobi equation

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**Definition 8** *We say that a map  $w : [0, T] \times \mathcal{M} \rightarrow \mathbb{R}$  is a sub-solution of the HJ equation (7) if  $w$  is upper semi-continuous and if, for any  $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$ , for any  $(p_t, p_\mu) \in D_\delta^+ w(t_0, \mu_0)$ , we have for any  $\delta > 0$ ,*

$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -C\delta \quad (8)$$

*where  $C > 0$  is a constant which depends only of  $\mathcal{H}$ .*

*In a similar way,  $w$  is a super-solution of the HJ equation (7) if  $w$  is lower semicontinuous and if, for any  $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$ , for any  $(p_t, p_\mu) \in D_\delta^- w(t_0, \mu_0)$ , we have*

$$p_t + \mathcal{H}(\mu_0, p_\mu) \leq C\delta . \quad (9)$$

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## Values and HJI Equations

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$$\mathcal{H}^+(\mu, p) = \inf_{u \in U} \sup_{v \in V} \int_{\mathbb{R}^N} \langle f(x, u, v), p(x) \rangle d\mu(x)$$

$$\mathcal{H}^-(\mu, p) = \sup_{v \in V} \inf_{u \in U} \int_{\mathbb{R}^N} \langle f(x, u, v), p(x) \rangle d\mu(x) .$$

**Lemma 9** *Let  $\mu, \nu \in \mathcal{M}$ ,  $\gamma$  be an optimal plan from  $\mu$  to  $\nu$ , and  $p \in L^2_\mu$  and  $q \in L^2_\nu$  be defined by (5) and (6) respectively. Then, for  $\mathcal{H} = \mathcal{H}^+$  or  $\mathcal{H} = \mathcal{H}^-$*

$$|\mathcal{H}(\mu, p) - \mathcal{H}(\nu, q)| \leq k(\mathbf{d}(\mu, \nu))^2 ,$$

$$\forall \varphi, \int_{\mathbb{R}^N} \langle \varphi(x), p(x) \rangle d\mu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x)$$

proof for  $\mathcal{H}^-$

$$\begin{aligned} \mathcal{H}^-(\mu, p) &= \sup_v \inf_u \int_{\mathbb{R}^N} \langle f(x, u, v), p(x) \rangle d\mu(x) \\ &= \sup_v \inf_u \int_{\mathbb{R}^{2N}} \langle f(x, u, v), x - y \rangle d\gamma(x, y) \\ &\leq \sup_v \inf_u \int_{\mathbb{R}^{2N}} \langle f(y, u, v), x - y \rangle d\gamma(x, y) \\ &\quad + k \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma(x, y) \\ &\leq \sup_v \inf_u \int_{\mathbb{R}^N} \langle f(y, u, v), q(y) \rangle d\nu(y) \\ &\quad + k \mathbf{d}^2(\mu, \nu) \\ &\leq \mathcal{H}^-(\nu, q) + k \mathbf{d}^2(\mu, \nu) \end{aligned}$$

□

**Proposition 10** *The upper value function  $V^+$  is a solution to HJI with  $\mathcal{H} := \mathcal{H}^+$  while the lower value function  $V^-$  is a solution to HJI with  $\mathcal{H} := \mathcal{H}^-$ .*

proof of  $V^+$  is a subsolution

**Fix**  $(t_0, \mu_0) \in (0, T) \times \mathcal{M}$ ,  $\delta > 0$  and  $(p_t, p_\mu) \in D_\delta^+ V^+(t_0, \mu_0)$ .

**We will prove that**

$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -\delta \quad (10)$$

**Consider**  $t \in (t_0, T)$ . **For any**  $\alpha \in \mathcal{A}(t_0)$  and  $\beta \in \mathcal{B}(t_0)$  **define**  $\varphi_{\alpha, \beta} \in \mathcal{C}_b(\mathbb{R}^N, \mathbb{R}^N)$  **such that**

$$(\text{id}_{\mathbb{R}^N} + \varphi_{\alpha, \beta})(x) = X_t^{t_0, x, \alpha, \beta} = x + \int_{t_0}^t f(x(s), u(s), v(s)) ds,$$

**where**  $(u, v)$  **is associated with**  $(\alpha, \beta)$  **and**  $x(s) = X_s^{t_0, x, \alpha, \beta}$ .

$$\begin{aligned}
& V^+(t, X_t^{t_0, \cdot, \alpha, v} \# \mu_0) - V^+(t_0, \mu_0) - p_t(t - t_0) \\
& - \int_{\mathbb{R}^N} < \int_{t_0}^t f(X_s^{t_0, x, \alpha, \beta}, u(s), v(s)) ds, p_\mu(x) > d\mu(x) \quad (11) \\
& \leq (\|\varphi_{\alpha, \beta}\|_{L_\mu^2} + |t - t_0|)(\varepsilon(t, \varphi_{\alpha, \beta}) + \delta)
\end{aligned}$$

where  $\varepsilon(t, \varphi_{\alpha, \beta}) \rightarrow 0$  as  $t \rightarrow t_0$  and  $\varphi_{\alpha, \beta} \rightarrow 0$  in  $L_\mu^2$ . Passing to the sup on  $v$  and inf on  $\alpha$ , we obtain by DDP

$$\begin{aligned}
0 \leq & \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{v \in \mathcal{V}(t_0)} \left[ \int_{\mathbb{R}^N} < \int_{t_0}^t f(X_s^{t_0, x, \alpha(v), v}, \alpha(v)(s), v(s)) ds, p_\mu(x) > d\mu \right. \\
& \left. + p_t(t - t_0) + (\|\varphi_{\alpha(v), v}\|_{L_\mu^2} + |t - t_0|)(\delta + \varepsilon(t, \varphi_{\alpha, \beta})) \right]
\end{aligned}$$

for  $\|\varphi_{\alpha(v), v}\|_{L_\mu^2} + |t - t_0|$  small enough.



For  $t$  close enough to  $t_0$  we obtain

$$0 \leq \inf_{u \in U} \sup_{v \in \mathcal{V}(t_0)} \left[ \int_{\mathbb{R}^N} < \int_{t_0}^t f(x, u, v(s)) ds, p_\mu(x) > d\mu(x) \right. \\ \left. + p_t(t - t_0) + (\|\varphi_{u,v}\|_{L_\mu^2} + |t - t_0|)(\delta + \varepsilon(t, \varphi_{u,v})) \right]$$

when we restrict the infimum to nonanticipative strategies  $\alpha$  which has constant control values. Hence

$$0 \leq \inf_{u \in U} \sup_{v \in V} \left[ (t - t_0) \int_{\mathbb{R}^N} < f(x, u, v) ds, p_\mu(x) > d\mu(x) \right. \\ \left. + p_t(t - t_0) + (\|\varphi_{u,v}\|_{L_\mu^2} + |t - t_0|)(\delta + \varepsilon(t, \varphi_{u,v})) \right]$$

Dividing this inequality by  $t - t_0$  and letting  $t \rightarrow t_0^+$  gives, since  $\|\varphi_{u,v}\|_{L_\mu^2} = O(t - t_0)$ ,

$$p_t + \mathcal{H}(\mu_0, p_\mu) \geq -(1 + M)\delta .$$



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## Comparison Principle for HJI

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$$w_t + \mathcal{H}(\mu, Dw) = 0$$

### Assumptions on $\mathcal{H}$

- $p \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N) \mapsto \mathcal{H}(\mu, \cdot)$  is positively homogeneous.
- for any  $\mu, \nu \in \mathcal{M}$ , if  $\gamma$  is the optimal plan from  $\mu$  to  $\nu$ , and  $p \in L^2_\mu$  and  $q \in L^2_\nu$  are defined by (5) and (6) respectively,

$$|\mathcal{H}(\mu, p) - \mathcal{H}(\nu, q)| \leq k(\mathbf{d}(\mu, \nu))^2. \quad (12)$$

$$\forall \varphi, \int_{\mathbb{R}^N} \langle \varphi(x), p(x) \rangle d\mu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(x), x - y \rangle d\gamma(x)$$

$$\forall \varphi, \int_{\mathbb{R}^N} \langle \varphi(x), q(x) \rangle d\nu(x) = \int_{\mathbb{R}^{2N}} \langle \varphi(y), x - y \rangle d\gamma(x)$$

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## Comparison principle for HJI

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**Theorem 11** *Let  $w_1$  be a bounded and Lipschitz continuous subsolution and  $w_2$  be a bounded and Lipschitz continuous supersolution to (7). Then*

$$\inf_{[0,T] \times \mathcal{M}} (w_2 - w_1) = \inf_{\mathcal{M}} w_2(T, \cdot) - w_1(T, \cdot) .$$

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## Proof of Comparison Principle

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$$A = \inf_{\mu \in \mathcal{M}} w_2(T, \mu) - w_1(T, \mu) .$$

Since  $\mathcal{H}$  is independant of  $w$ ,  $w_1 - A$  is still a subsolution.  
So we suppose without loss of generality that  $A = 0$ .

**By Contradiction**

$$-\xi := \inf_{\mu \in \mathcal{M}, t \in [0, T]} w_2(t, \mu) - w_1(t, \mu) < 0 .$$

**And choose  $(t_0, \mu_0) \in [0, T] \times \mathcal{M}$  such that**

$$(w_2 - w_1)(t_0, \mu_0) < -\xi/2. \tag{13}$$

Let  $C > 0$  such that  $\forall \delta > 0$

$$\forall (p_t, p_\mu) \in D_\delta^+ w_1(t_0, \mu_0), p_t + \mathcal{H}(\mu_0, p_\mu) \geq -C\delta$$

$$\forall (p_t, p_\mu) \in D_\delta^- w_2(t_0, \mu_0) p_t + \mathcal{H}(\mu_0, p_\mu) \leq C\delta .$$

Fix  $\varepsilon > 0$ ,  $\eta > 0$  and  $\delta > 0$  sufficiently small such that

$$\xi > 2\eta T + \frac{k^2\varepsilon}{2} \quad \text{and} \quad 2C\delta + 2k(\delta + k)^2\varepsilon < \eta . \quad (14)$$

We consider the following continuous function defined on  $([0, T] \times \mathcal{M})^2$ :

$$\Phi(s, \mu, t, \nu) = -w_1(s, \mu) + w_2(t, \nu) + \frac{1}{\varepsilon} (\mathbf{d}^2(\mu, \nu) + (t - s)^2) - \eta s .$$

**Define**

$$(\bar{\mu}, \bar{\nu}, \bar{s}, \bar{t}) \in \text{Arg} \min_{[0, T] \times \mathcal{M}} \Phi$$

From Ekeland Variational Principle that  $\exists(\bar{\mu}, \bar{\nu}, \bar{s}, \bar{t}) \in \mathcal{M}^2 \times [0, T]^2$  such that for any  $(s, \mu, t, \nu) \in ([0, T] \times \mathcal{M})^2$

$$\left\{ \begin{array}{l} i) \quad \Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(t_0, \mu_0, t_0, \mu_0) \\ ii) \quad \Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(s, \mu, t, \nu) \\ \quad \quad + \delta([\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} + [\mathbf{d}^2(\nu, \bar{\nu}) + |t - \bar{t}|^2]^{\frac{1}{2}}) \end{array} \right. \quad (15)$$

**CLAIM 1**  $\rho^2 := \mathbf{d}^2(\bar{\mu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2 \leq (k + \delta)^2 \varepsilon^2$  **Indeed,**

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(\bar{s}, \bar{\mu}, \bar{s}, \bar{\mu}) + \delta[\mathbf{d}^2(\bar{\nu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2]^{\frac{1}{2}}$$

**Since  $w_2$  is  $k$ -Lipschitz continuous,**

$$\begin{aligned} \delta\rho + w_2(\bar{s}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - \eta\bar{s} &\geq w_2(\bar{t}, \bar{\nu}) - w_1(\bar{s}, \bar{\mu}) + \frac{1}{\varepsilon}\rho^2 - \eta\bar{s} \\ &\geq w_2(\bar{s}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - k\rho + \frac{1}{\varepsilon}\rho^2 - \eta\bar{s} \end{aligned}$$

**Hence  $\rho \leq (k + \delta)\varepsilon$ ,**

Assume  $\bar{s}, \bar{t} \in (0, T)$ . Let  $\gamma$  be the optimal transport plan between  $\bar{\mu}$  and  $\bar{\nu}$  and  $p \in L^2(\bar{\mu})$  and  $q \in L^2(\bar{\nu})$  associated.

**CLAIM 2**

$$\left( \frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta, \frac{2}{\varepsilon}p \right) \in D_{\delta}^{+} w_1(\bar{s}, \bar{\mu}), \quad \left( \frac{2}{\varepsilon}(\bar{s} - \bar{t}), \frac{2}{\varepsilon}q \right) \in D_{\delta}^{-} w_2(\bar{t}, \bar{\nu}),$$

From (15)-ii), we have for any  $(s, \mu)$ ,

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(s, \mu, \bar{t}, \bar{\nu}) + \delta[\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}}.$$

$$\begin{aligned} w_1(s, \mu) &\leq w_1(\bar{s}, \bar{\mu}) + \frac{1}{\varepsilon}(\mathbf{d}^2(\mu, \bar{\nu}) - \mathbf{d}^2(\bar{\mu}, \bar{\nu}) + (s - \bar{t})^2 - (\bar{s} - \bar{t})^2) \\ &\quad + \delta[\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} + \eta(\bar{s} - s) \end{aligned} \tag{16}$$

We will replace  $\mu$  by  $(\text{id}_{\mathbb{R}^N} + \varphi)\# \bar{\mu}$  with  $\varphi \in \mathcal{C}_b(\mathbb{R}^N, \mathbb{R}^N)$



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## Estimation

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**Observe  $\gamma' = (\text{id}_{\mathbb{R}^N} + \varphi, \text{id}_{\mathbb{R}^N})\#\gamma$  is an admissible transport plan from  $(\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}$  to  $\bar{\nu}$ . Hence**

$$\begin{aligned} \mathbf{d}^2((\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}, \bar{\nu}) - \mathbf{d}^2(\bar{\mu}, \bar{\nu}) &\leq \\ \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma'(x, y) - \int_{\mathbb{R}^{2N}} |x - y|^2 d\gamma(x, y) & \\ = \int_{\mathbb{R}^{2N}} (|x + \varphi(x) - y|^2 - |x - y|^2) d\gamma(x, y) & \\ = \int_{\mathbb{R}^{2N}} (2 \langle x - y, \varphi(x) \rangle + |\varphi(x)|^2) d\gamma(x, y) & \\ = \int_{\mathbb{R}^N} \langle 2p(x), \varphi(x) \rangle d\bar{\mu}(x) + \|\varphi\|_{L^2_{\bar{\mu}}}^2 \quad (\text{from the def. of } p) & \end{aligned}$$

$$\begin{aligned} w_1(s, (\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}) &\leq w_1(\bar{s}, \bar{\mu}) + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \langle 2p(x), \varphi(x) \rangle d\bar{\mu}(x) + \|\varphi\|_{L^2_{\bar{\mu}}}^2 \\ &+ \frac{1}{\varepsilon} (s - \bar{s}) [(s - \bar{s}) + 2(\bar{s} - \bar{t})] + \delta [\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} + \eta(\bar{s} - s) \end{aligned}$$

**So**

$$\begin{aligned} & w_1(s, (\text{id}_{\mathbb{R}^N} + \varphi)\#\bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - (s - \bar{s})\left[\frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta\right] \\ & - \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \langle 2p(x), \varphi(x) \rangle d\bar{\mu}(x) + \|\varphi\|_{L^2_{\bar{\mu}}}^2 \leq \\ & + \frac{1}{\varepsilon}(s - \bar{s})^2 + \delta[\mathbf{d}^2(\mu, \bar{\mu}) + |s - \bar{s}|^2]^{\frac{1}{2}} \end{aligned}$$

**Hence**

$$\left( \frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta, \frac{2}{\varepsilon}p \right) \in D_{\delta}^+ w_1(\bar{s}, \bar{\mu})$$

**which the claim 2.**

---

By Claim 2, because  $w_1$  sub solution,  $w_2$  supersolution

$$\frac{2}{\varepsilon}(\bar{s} - \bar{t}) - \eta + \mathcal{H}\left(\bar{\mu}, \frac{2}{\varepsilon}p\right) \geq -C\delta .$$

$$\frac{2}{\varepsilon}(\bar{s} - \bar{t}) + \mathcal{H}\left(\bar{\mu}, \frac{2}{\varepsilon}q\right) \leq C\delta .$$

Subtracting we obtain  $\mathcal{H}(\bar{\nu}, q) - \mathcal{H}(\bar{\mu}, p) \leq 2C\delta - \eta$ .

But from the assumption on  $\mathcal{H}$  we have  $\mathcal{H}(\bar{\nu}, q) - \mathcal{H}(\bar{\mu}, p) \geq -k\mathbf{d}^2(\bar{\mu}, \bar{\nu})$ . So

$$-k(k + \delta)^2\varepsilon \leq 2C\delta - \eta ,$$

a contradiction with (14).

We have to check that  $\bar{s}$  and  $\bar{t}$  cannot be equal to 0 or  $T$ . Let us assume for instance that  $\bar{s} = T$ . We first note that

$$\Phi(t_0, \mu_0, t_0, \mu_0) = w_2(t_0, \mu_0) - w_1(t_0, \mu_0) - \eta t_0 \leq -\xi/2$$

From (Ekeland), i), we have

$$\Phi(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \Phi(t_0, \mu_0, t_0, \mu_0) \leq -\xi/2 .$$

Since  $\bar{s} = T$  and  $w_2$  is  $k$ -Lipschitz continuous, we get (setting as before  $\rho := [\mathbf{d}^2(\bar{\nu}, \bar{\nu}) + |\bar{s} - \bar{t}|^2]^{\frac{1}{2}}$ )

$$\begin{aligned} -\xi/2 &\geq w_2(\bar{t}, \bar{\nu}) - w_1(T, \bar{\mu}) + \frac{1}{\varepsilon}\rho^2 - \eta T \\ &\geq w_2(T, \bar{\mu}) - w_1(T, \bar{\mu}) - k\rho + \frac{1}{\varepsilon}\rho^2 - \eta T \\ &\geq -k\rho + \frac{1}{\varepsilon}\rho^2 - \eta T , \end{aligned}$$

A contradiction with the choices of  $\eta, \delta, \varepsilon$  and the estimation of  $\rho$ .

To show that  $\bar{s} \neq 0$  and  $\bar{t} \neq 0$ , it is enough to use the standard fact that  $w_1$  and  $w_2$  are respectively sub- and supersolutions up to  $t = 0$ ,

**Lemma 12** *If  $w$  is a subsolution (resp. a supersolution) of (7) on the time interval  $(0, T)$ , then  $w$  is also a subsolution (resp. a supersolution) on  $[0, T)$ .*

□

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## Uniqueness Result for HJI

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**Corollary 13** *There exists at most one lipschitz continuous solution of (7) satisfying*

$$w(T, \mu) = \int_{\mathbb{R}^N} g(x) d\mu(x) , \forall \mu \in \mathcal{M}.$$

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## Existence of A value

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**Theorem 14** *We suppose the following Isaacs condition:*

$$\mathcal{H}^+ = \mathcal{H}^-.$$

*Then the game has a value. Namely:*

$$V^+(t, \mu) = V^-(t, \mu) \quad \forall (t, \mu) \in [0, T] \times \mathcal{M}.$$

*Furthermore  $V^+ = V^-$  is the unique solution of the Hamilton-Jacobi equation (7) with  $\mathcal{H} = \mathcal{H}^+ = \mathcal{H}^-$ .*

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**Thank You for your Attention**

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