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# COMPUTING LOW-DEGREE ISOGENIES IN GENUS 2 WITH THE DOLGACHEV–LEHAVI METHOD

BENJAMIN SMITH

ABSTRACT. Let  $\ell$  be a prime, and  $\mathcal{H}$  a curve of genus 2 over a field  $\mathbb{k}$  of characteristic not 2 or  $\ell$ . If  $S$  is a maximal Weil-isotropic subgroup of  $\mathcal{J}_{\mathcal{H}}[\ell]$ , then  $\mathcal{J}_{\mathcal{H}}/S$  is isomorphic to the Jacobian  $\mathcal{J}_{\mathcal{X}}$  of some (possibly reducible) curve  $\mathcal{X}$ . We investigate the Dolgachev–Lehavi method for constructing the curve  $\mathcal{X}$ , simplifying their approach and making it more explicit. The result, at least for  $\ell = 3$ , is an efficient and easily programmable algorithm suitable for number-theoretic calculations.

## 1. INTRODUCTION

Let  $\ell \geq 3$  be prime, and let  $\mathcal{H}$  be a curve of genus 2 over a perfect field  $\mathbb{k}$  of characteristic not 2 or  $\ell$ . Let  $\mathcal{J}_{\mathcal{H}}$  be the Jacobian of  $\mathcal{H}$ , and let  $S$  be a maximal  $\ell$ -Weil isotropic subgroup of  $\mathcal{J}_{\mathcal{H}}[\ell]$ ; since  $\ell$  is prime,  $S \cong (\mathbb{Z}/\ell\mathbb{Z})^2$ . The quotient  $\mathcal{J}_{\mathcal{H}}/S$  is isomorphic (as a principally polarized abelian variety) to a Jacobian  $\mathcal{J}_{\mathcal{X}}$ , where  $\mathcal{X}$  is some curve of genus 2 (see [14]); hence, there exists an isogeny

$$\phi: \mathcal{J}_{\mathcal{H}} \rightarrow \mathcal{J}_{\mathcal{X}}$$

with kernel  $S$  (note that  $\mathcal{X}$  may be reducible, in which case  $\mathcal{J}_{\mathcal{X}}$  is a product of elliptic curves). Our aim is to compute an explicit form for  $\mathcal{X}$  given  $\mathcal{H}$  and  $S$ .

In the case  $\ell = 2$ , the problem is resolved by the well-known Richelot construction (see [3] and [5, Chapter 9]). More generally, if  $\mathbb{k}$  is finite, then we can apply the explicit theta function-based algorithms of Lubicz and Robert [10], implemented in the freely-available `avIsogenies` package [1].

Alternatively, there is the algebraic-geometric approach described by Dolgachev and Lehavi [7], which computes the image of the theta divisor on  $\mathcal{J}_{\mathcal{H}}$  in the Kummer surface of  $\mathcal{J}_{\mathcal{X}}$ . As presented in [7], this approach has two drawbacks:

- (1) it is not effective for  $\ell \neq 3$ , and
- (2) for  $\ell = 3$ , where theta structures are involved, it assumes  $\mathbb{k} \subset \mathbb{C}$ .

In this work we render the kernel of the Dolgachev–Lehavi method completely explicit, with a view to computations in number theory. Our intention is to provide a sort of “user’s guide” to the algorithm and its concrete implementation. For  $\ell = 3$  we obtain a simple, efficient, and easily-programmable algorithm (that does not require  $\mathbb{k} \subset \mathbb{C}$ ). Our algorithm retains the pleasing geometric flavour of the original, but is better-suited to everyday calculations.

## 2. AN OVERVIEW OF THE DOLGACHEV–LEHAVI CONSTRUCTION

We begin by briefly recalling the Dolgachev–Lehavi construction, before treating it in detail in the following sections. Suppose  $\mathcal{H}/\mathbb{k}$ ,  $S$ ,  $\phi$ , and  $\mathcal{X}$  are as above; we assume we are given an explicit form for  $\mathcal{H}$  and  $S$ , and we want to compute an explicit form for  $\mathcal{X}$ . Dolgachev and Lehavi observe that if  $\Theta_{\mathcal{H}}$  and  $\Theta_{\mathcal{X}}$  are theta divisors on  $\mathcal{J}_{\mathcal{H}}$  and  $\mathcal{J}_{\mathcal{X}}$ , respectively, then  $\phi(\Theta_{\mathcal{H}})$  is in  $|\ell\Theta_{\mathcal{X}}|$  (see [7, Proposition 2.4]); and as such, the image of  $\phi(\Theta_{\mathcal{H}})$  in the Kummer surface  $\mathcal{K}_{\mathcal{X}} = \mathcal{J}_{\mathcal{X}}/\langle \pm 1 \rangle$  is a degree- $2\ell$  rational curve<sup>1</sup> in  $\mathbb{P}^3$  of

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<sup>1</sup>By “rational curve” we mean a curve of genus 0. In all other contexts, “rational” means “defined over  $\mathbb{k}$ ”.

arithmetic genus  $(\ell^2 - 1)/2$  and with  $(\ell^2 - 1)/2$  ordinary double points corresponding to the nonzero elements of  $S$ , up to sign [7, Proposition 3.1]. We can compute this curve *without* knowing  $\phi$  by expressing the map  $\Phi: \mathcal{H} \cong \Theta_{\mathcal{H}} \subset \mathcal{I}_{\mathcal{H}} \rightarrow \mathcal{I}_{\mathcal{X}} \rightarrow \mathcal{K}_{\mathcal{X}} \subset \mathbb{P}^3$  as the composition of a double cover  $\rho_{2\ell}$  of a rational normal curve in  $\mathbb{P}^{2\ell}$  with a projection  $\pi: \mathbb{P}^{2\ell} \rightarrow \mathbb{P}^3$  whose centre depends on certain secants corresponding to the nonzero elements of  $S$ , up to sign. The images under  $\Phi$  of the Weierstrass points of  $\mathcal{H}$  lie on a conic  $\mathcal{Q}$  contained in a hyperplane of  $\mathbb{P}^3$ ; that is, a trope of  $\mathcal{K}_{\mathcal{X}}$ . The double cover of  $\mathcal{Q}$  ramified over the Weierstrass point images is then (a quadratic twist of)  $\mathcal{X}$ .

### 3. THE DOMAIN CURVE

We suppose that  $\mathcal{H}/\mathbb{k}$  is presented as a nonsingular projective model

$$(1) \quad \mathcal{H}: Y^2 = F(X, Z) = \sum_{i=0}^6 F_i X^i Z^{6-i} \subset \mathbb{P}(1, 3, 1),$$

where  $F$  is a squarefree homogeneous sextic over  $\mathbb{k}$  (such a model always exists when  $\mathbb{k}$  is perfect and has characteristic not 2: see [5, §1.3]). The hyperelliptic involution of  $\mathcal{H}$  is

$$\iota_{\mathcal{H}}: (X : Y : Z) \longmapsto (X : -Y : Z).$$

The divisor at infinity on  $\mathcal{H}$  is

$$D_{\infty} = (1 : \sqrt{F_6} : 0) + (1 : -\sqrt{F_6} : 0);$$

we observe that  $D_{\infty}$  is defined over  $\mathbb{k}$ , fixed by  $\iota_{\mathcal{H}}$ , and equal to  $2(1 : 0 : 0)$  if  $F_6 = 0$ .

The six Weierstrass points of  $\mathcal{H}$  are the fixed points of  $\iota_{\mathcal{H}}$ ; they correspond to the projective roots of the sextic  $F$ . The Weierstrass divisor  $W_{\mathcal{H}}$  of  $\mathcal{H}$  is the effective divisor cut out by  $Y = 0$ ; if  $F(X, Z) = \prod_{i=1}^6 (z_i X - x_i Z)$  over  $\overline{\mathbb{k}}$ , then

$$W_{\mathcal{H}} = (x_1 : 0 : z_1) + \cdots + (x_6 : 0 : z_6).$$

Note that  $W_{\mathcal{H}}$  is defined over  $\mathbb{k}$ . Finally, we fix a canonical divisor on  $\mathcal{H}$ , defining

$$K_{\mathcal{H}} = W_{\mathcal{H}} - 2D_{\infty}.$$

### 4. THE KERNEL OF THE ISOGENY

When defining their method for  $\ell = 3$ , Dolgachev and Lehavi state “unfortunately, we do not know how to input explicitly the pair  $(\mathcal{H}, S)$ . Instead we consider  $\mathcal{H}$  with an odd theta structure.” We will take a rather more middlebrow approach to the problem: we suppose that  $\mathcal{H}$  is presented in the form (1), and that  $S$  is given as a collection of divisor classes on  $\mathcal{H}$  expressed using an extended Mumford representation (detailed below).

Our motivation for this choice is simple: this is precisely how one computes with hyperelliptic Jacobians in computational algebra systems such as Magma [11, 2] and Sage [13]. This choice also radically simplifies the algorithm: we can omit the theta structure calculations and pass directly to the secant computations (short-circuiting the first four steps of the algorithm in [7, §5.1]).

Points on  $\mathcal{I}_{\mathcal{H}}$  correspond to divisor classes of degree zero on  $\mathcal{H}$ . The Riemann–Roch theorem tells us that every nontrivial degree-0 class has a unique representative in the form  $P + Q - D_{\infty}$  (this representation fails to be unique for the trivial class, because  $[P + \iota_{\mathcal{H}}(P) - D_{\infty}] = 0$  for every  $P$  in  $\mathcal{H}(\overline{\mathbb{k}})$ ).

Let  $e$  be a point of  $\mathcal{I}_{\mathcal{H}}$ , corresponding to a divisor class  $[P + Q - D_{\infty}]$ . The effective divisor  $P + Q$  is cut out by an ideal in the form  $(A(X, Z), Y - B(X, Z))$ , where  $B$  is a homogeneous cubic and  $A$  a homogeneous polynomial of degree  $d \leq 2$ . The triple

$$\langle a(x), b(x), d \rangle := \langle A(x, 1), B(x, 1), d \rangle$$

then encodes the point  $e$  (with the convention that  $A$  is chosen such that  $a$  is monic). Note that if  $\mathbb{k}'$  is an extension of  $\mathbb{k}$ , then  $e = \langle a, b, d \rangle$  is in  $\mathcal{I}_{\mathcal{H}}(\mathbb{k}')$  if and only if  $a$  and  $b$  have coefficients in  $\mathbb{k}'$ .

Conversely, given a triple  $\langle a, b, d \rangle$ , we recover the corresponding point of  $\mathcal{J}_{\mathcal{H}}$  by computing the effective divisor cut out by  $(A(X, Z), Y - B(X, Z))$ , where  $B$  is the degree-3 homogenization of  $b$  and  $A$  is the degree- $d$  homogenization of  $a$ , and then subtracting  $(d/2)D_{\infty}$ . If  $\mathcal{H}$  has two points at infinity (that is, if  $F_6 \neq 0$ ) then  $d$  must be either 2 or 0. In the case where  $\mathcal{H}$  has a single point at infinity (that is, when  $F_6 = 0$ ) we always have  $d = \deg a$ , and the pair  $\langle a, b \bmod a \rangle$  is the standard Mumford representation. The advantage of the extended representation above is that it gracefully handles the general case where there are two points at infinity.

*Example 4.1.* Consider the following points on the Jacobian of  $\mathcal{H}: Y^2 = X^6 - Z^6$ .

- 0 is represented by  $(1, 0, 0)$ .
- $[(1 : 0 : 1) + (-1 : 0 : 1) - D_{\infty}]$  is represented by  $\langle x^2 - 1, 0, 2 \rangle$ .
- $[(1 : 1 : 0) - (1 : -1 : 0)] = [2(1 : 1 : 0) - D_{\infty}]$  is represented by  $\langle 1, x^3, 2 \rangle$ .

In this article, we will assume that the points of  $S$  are all  $\mathbb{k}$ -rational. This simplifies the exposition and the computations; however, all of our calculations are symmetric in the elements of  $S$ . The algorithm should therefore be easily adapted to the case where  $S$  is rational but its elements are not.

## 5. THE RATIONAL NORMAL CURVE

The Riemann–Roch space  $L(2\ell K_{\mathcal{H}})$  is a direct sum of subspaces

$$L(2\ell K_{\mathcal{H}}) = L(2\ell K_{\mathcal{H}})^+ \oplus L(2\ell K_{\mathcal{H}})^-,$$

where  $\iota_{\mathcal{H}}$  acts as  $+1$  on the elements of  $L(2\ell K_{\mathcal{H}})^+$  and  $-1$  on the elements of  $L(2\ell K_{\mathcal{H}})^-$ . Writing  $x = X/Z$  and  $y = Y/Z^3$ , we have

$$L(2\ell K_{\mathcal{H}})^+ = \langle x^i / y^{2\ell} \rangle_{i=0}^{2\ell} \quad \text{and} \quad L(2\ell K_{\mathcal{H}})^- = \langle x^i / y^{2\ell-1} \rangle_{i=0}^{2\ell-3}.$$

The space  $L(2\ell K_{\mathcal{H}})^+$  corresponds to the linear system  $|2\ell K_{\mathcal{H}}|^{\langle \iota_{\mathcal{H}} \rangle}$ ; we see immediately that it is  $(2\ell + 1)$ -dimensional, and therefore defines a map

$$\rho_{2\ell} : \mathcal{H} \longrightarrow \mathcal{R}_{2\ell} \subset \mathbb{P}^{2\ell}$$

onto a curve  $\mathcal{R}_{2\ell}$  in  $\mathbb{P}^{2\ell}$ . Fixing coordinates on  $\mathbb{P}^{2\ell}$ , we take  $\rho_{2\ell}$  to be defined by

$$\rho_{2\ell} : (X : Y : Z) \longmapsto (U_0 : \cdots : U_{2\ell}) = (X^0 Z^{2\ell} : X Z^{2\ell-1} : \cdots : X^{2\ell-1} Z : X^{2\ell}).$$

We see that  $\mathcal{R}_{2\ell}$  is a rational normal curve of degree  $2\ell$  in  $\mathbb{P}^{2\ell}$ , and  $\rho_{2\ell}$  is a double cover:

$$(2) \quad \rho_{2\ell}(P) = \rho_{2\ell}(Q) \iff (P = Q \text{ or } P = \iota_{\mathcal{H}}(Q)).$$

(Essentially,  $\rho_{2\ell}$  is a composition of the canonical map of  $\mathcal{H}$  and an  $\ell$ -uple embedding.)

## 6. THE SECANT LINES

We adopt the following convention: if  $S$  is a set of points in some projective space  $\mathbb{P}^n$ , then  $\langle S \rangle$  denotes the linear subspace of  $\mathbb{P}^n$  generated by  $S$ .

For any pair of points  $P$  and  $Q$  on  $\mathcal{H}$ , we define  $\mathcal{L}_{P,Q}$  to be the line in  $\mathbb{P}^{2\ell}$  intersecting  $\mathcal{R}_{2\ell}$  in  $\rho_{2\ell}(P) + \rho_{2\ell}(Q)$ ; that is,

$$\mathcal{L}_{P,Q} := \begin{cases} \langle \rho_{2\ell}(P), \rho_{2\ell}(Q) \rangle & \text{if } P \notin \{Q, \iota_{\mathcal{H}}(Q)\} \\ T_{\rho_{2\ell}(P)}(\mathcal{R}_{2\ell}) & \text{otherwise.} \end{cases}$$

We can also define secant lines corresponding to nonzero Jacobian elements: if  $e$  is a nonzero point on  $\mathcal{J}_{\mathcal{H}}$ , then we define

$$\mathcal{L}_e := \mathcal{L}_{P,Q} \quad \text{where } e = [P + Q - D_{\infty}].$$

Observe that  $\mathcal{L}_{P,Q} = \mathcal{L}_{\iota_{\mathcal{H}}(P),Q} = \mathcal{L}_{\iota_{\mathcal{H}}(P),Q} = \mathcal{L}_{\iota_{\mathcal{H}}(P),\iota_{\mathcal{H}}(Q)}$  for all  $P$  and  $Q$  on  $\mathcal{H}$ , so

$$\mathcal{L}_e = \mathcal{L}_{-e}$$

for all  $e$  in  $\mathcal{J}_{\mathcal{H}} \setminus \{0\}$ .

*Remark 6.1.* Dolgachev and Lehavi define secants  $l_e = \langle \rho_{2\ell}(P_1), \rho_{2\ell}(P_2) \rangle$  for each non-trivial point  $e = [P_1 - P_2]$  in  $\mathcal{J}_{\mathcal{H}}$  (see [7, Theorem 1.1]). Our  $\mathcal{L}_e$  is equal to  $l_e$ , because  $[P_1 - P_2] = [P_1 + \iota_{\mathcal{H}}(P_2) - D_{\infty}]$  and  $\mathcal{L}_{P_1, P_2} = \mathcal{L}_{P_1, \iota_{\mathcal{H}}(P_2)}$ .

The following lemma gives explicit and rational formulæ for the secants  $\mathcal{L}_e$  and their intersections with arbitrary hyperplanes in  $\mathbb{P}^2$ . These formulæ are central to the explicit Dolgachev–Lehavi method.

**Lemma 6.2.** *Let  $e = \langle a, b, d \rangle$  be a nonzero point of  $\mathcal{J}_{\mathcal{H}}$ . Let  $H : \sum_{i=0}^{2\ell} H_i U_i = 0$  be a hyperplane in  $\mathbb{P}^{2\ell}$ , and write  $h(x) := \sum_{i=0}^{2\ell} H_i x^i$ .*

- (1) *If  $a = 1$  and  $d = 2$ , then*

$$\mathcal{L}_e = \langle (0 : \cdots : 0 : 1 : 0), (0 : \cdots : 0 : 0 : 1) \rangle.$$

- (a) *If  $H_{2\ell} = H_{2\ell-1} = 0$ , then  $\mathcal{L}_e \subset H$ .*  
 (b) *Otherwise,  $\mathcal{L}_e \cap H = (0 : \cdots : 0 : H_{2\ell} : -H_{2\ell-1})$ .*

- (2) *If  $a(x) = x - \alpha$ , then*

$$\mathcal{L}_e = \langle (0 : \cdots : 0 : 1), (1 : \cdots : \alpha^{2\ell}) \rangle.$$

- (a) *If  $h(\alpha) = 0$  and  $H_{2\ell} = 0$ , then  $\mathcal{L}_e \subset H$ .*  
 (b) *Otherwise,  $\mathcal{L}_e \cap H = (H_{2\ell} : H_{2\ell}\alpha : \cdots : H_{2\ell}\alpha^{2\ell-1} : H_{2\ell}\alpha^{2\ell} - h(\alpha))$ .*

- (3) *If  $a(x) = x^2 + a_1 x + a_0$  with  $a_1^2 \neq 4a_0$ , then*

$$\mathcal{L}_e = \langle (\pi_0 : \cdots : \pi_{2\ell}), (-a_1 : a_2 \pi_0 : a_2 \pi_1 : \cdots : a_2 \pi_{2\ell-1}) \rangle$$

where  $\pi_0 = 2$ ,  $\pi_1 = -a_1$ , and  $\pi_i = -a_1 \pi_{i-1} - a_2 \pi_{i-2}$  for  $i > 1$ .

- (a) *If  $a(x)$  divides  $h(x)$ , then  $\mathcal{L}_e \subset H$ .*  
 (b) *Otherwise,  $\mathcal{L}_e \cap H = (\gamma_0 : \cdots : \gamma_{2\ell})$  where*

$$\gamma_i = \sum_{0 \leq j \leq 2\ell} H_j (a_2^j \sigma_{i-j} - a_2^i \sigma_{j-i})$$

with  $\sigma_k = 0$  for  $k < 1$ ,  $\sigma_1 = 1$ , and  $\sigma_k = -a_1 \sigma_{k-1} - a_2 \sigma_{k-2}$  for  $k > 1$ .

- (4) *If  $a(x) = x^2 + a_1 x + a_0$  with  $a_1^2 = 4a_0$ , then writing  $\alpha$  for  $-a_1/2$ , we have*

$$\mathcal{L}_e = \langle (1 : \alpha : \cdots : \alpha^{2\ell}), (0 : 1 : 2\alpha : \cdots : 2\ell \alpha^{2\ell-1}) \rangle.$$

- (a) *If  $a(x)$  divides  $h(x)$ , then  $\mathcal{L}_e \subset H$ .*  
 (b) *Otherwise,  $\mathcal{L}_e \cap H = (\gamma_0 : \cdots : \gamma_{2\ell})$  where  $\gamma_i = i \alpha^{i-1} h(\alpha) - \alpha^i h'(\alpha)$ .*

*Proof.* In general, given points  $\alpha = (\alpha_0 : \cdots : \alpha_{2\ell})$  and  $\beta = (\beta_0 : \cdots : \beta_{2\ell})$  in  $\mathbb{P}^{2\ell}$ , we have

$$H \cap \mathcal{L}_{\alpha, \beta} = (A\beta_0 - B\alpha_0 : \cdots : A\beta_{2\ell} - B\alpha_{2\ell})$$

where  $A = \sum_{i=0}^{2\ell} H_i \alpha_i$  and  $B = \sum_{i=0}^{2\ell} H_i \beta_i$ ; if  $A = B = 0$ , then  $\mathcal{L}_{\alpha, \beta} \subset H$  (and the point above is not defined). In the following, we suppose  $e = [P + Q - D_{\infty}]$ ; we have  $e \neq 0$ , so we can suppose  $P \neq \iota_{\mathcal{H}}(Q)$ .

In case (1) both  $P$  and  $Q$  are at infinity, so  $\rho_{2\ell}(P) = \rho_{2\ell}(Q) = (0 : \cdots : 0 : 1)$ ; our expression for  $\mathcal{L}_e$  gives generators for the tangent to  $\mathcal{R}_{2\ell}$  at  $(0 : \cdots : 0 : 1)$ . The intersection formula follows immediately.

In case (2), we have  $P = (1 : 0 : 0)$  and  $Q = (\alpha : \pm \sqrt{F(\alpha, 1)} : 1)$ , so  $\rho_{2\ell}(P) = (0 : \cdots : 0 : 1)$  and  $\rho_{2\ell}(Q) = (1 : \alpha : \cdots : \alpha^{2\ell})$ . The intersection formula follows immediately.

In case (3), we have  $P = (\alpha : \pm \sqrt{F(\alpha, 1)} : 1)$  and  $Q = (\beta : \pm \sqrt{F(\beta, 1)} : 1)$  with  $\alpha \neq \beta$ ,  $\alpha + \beta = -a_1$ , and  $\alpha\beta = a_2$ ; so  $\rho_{2\ell}(P) = (1 : \alpha : \cdots : \alpha^{2\ell})$  and  $\rho_{2\ell}(Q) = (1 : \beta : \cdots : \beta^{2\ell})$ . If we take

$$T = (2 : \alpha + \beta : \cdots : \alpha^{2\ell} + \beta^{2\ell}) \quad \text{and} \quad S = (\alpha + \beta : 2\beta\alpha : \cdots : \alpha\beta^{2\ell} + \beta\alpha^{2\ell}),$$

then we easily verify that  $\mathcal{L}_{P, Q} = \mathcal{L}_{T, S}$ ; it is a straightforward exercise with symmetric polynomials to show that  $\alpha^i + \beta^i = \pi_i$  for  $0 \leq i \leq 2\ell$  and  $\alpha\beta^i + \beta\alpha^i = a_2\pi_{i-1}$  for  $i > 0$ , whence our formula for  $\mathcal{L}_e$ . The intersection  $H \cap \mathcal{L}_e$  is

$$H \cap \mathcal{L}_{P, Q} = (h(\alpha) - h(\beta) : h(\alpha)\beta - h(\beta)\alpha : \cdots : h(\alpha)\beta^{2\ell} - h(\beta)\alpha^{2\ell});$$

it is another straightforward exercise to show that

$$\alpha^j \beta^i - \beta^j \alpha^i = (\beta - \alpha)(a_2^j \sigma_{i-j} - a_2^i \sigma_{j-i}),$$

so  $h(\alpha)\beta^i - h(\beta)\alpha^i = \sum_{j=0}^{2\ell} H_j(\beta - \alpha)(a_2^j \sigma_{i-j} - a_2^i \sigma_{j-i}) = (\beta - \alpha)\gamma_i$  for  $0 \leq i \leq 2\ell$ , and thus  $H \cap \mathcal{L}_e = (\gamma_0 : \cdots : \gamma_{2\ell})$ .

In case (4), we have  $P = Q = (\alpha : \pm\sqrt{F(\alpha, 1)} : 1)$ ; our expression for  $\mathcal{L}_e$  gives generators for the tangent to  $\mathcal{R}_{2\ell}$  at  $\rho_{2\ell}(P) = (1 : \alpha : \cdots : \alpha^{2\ell})$ . The intersection formula follows.  $\square$

## 7. THE WEIERSTRASS SUBSPACE

Since  $\mathcal{R}_{2\ell}$  is a rational normal curve of degree  $2\ell$ , any  $2\ell + 1$  distinct points on  $\mathcal{R}_{2\ell}$  are linearly independent. In particular, the images of the six Weierstrass points of  $\mathcal{H}$  under  $\rho_{2\ell}$  are linearly independent because  $\ell \geq 3$ . In view of (2) the images are distinct, so the subspace

$$W := \langle \rho_{2\ell}(W_{\mathcal{H}}) \rangle \subset \mathbb{P}^{2\ell}$$

is five-dimensional.

For each  $0 \leq i \leq 2\ell - 6$ , we define a linear form

$$W_i := \sum_{j=0}^6 F_j U_{i+j}.$$

**Lemma 7.1.** *The space  $W$  is*

$$W = \bigcap_{i=0}^{2\ell-6} V(W_i) = V(\{W_i : 0 \leq i \leq 2\ell - 6\}).$$

*Proof.* Each hyperplane  $V(W_i)$  contains  $W$ , since  $W_i \circ \rho_{2\ell} = X^i Z^{2\ell-6-i} F(X, Z)$ . But the  $W_i$  are linearly independent, so the intersection  $\bigcap_{i=0}^{2\ell-6} V(W_i)$  is 5-dimensional, and hence equal to  $W$ .  $\square$

## 8. THE THEOREM OF DOLGACHEV AND LEHAVI

We are now ready to state the main theorem behind the Dolgachev–Lehavi method.

**Theorem 8.1** ([7, Theorem 1.1]). *There exists a unique hyperplane  $H \subset \mathbb{P}^{2\ell}$  such that*

- (1)  *$H$  contains  $W$ , and*
- (2) *the intersection points of  $H$  with the secants  $\mathcal{L}_e$  for each nonzero  $e$  in  $S$  are contained in a subspace  $N$  of codimension 3 in  $H$ .*

*The image of the Weierstrass divisor under the projection  $\mathbb{P}^{2\ell} \rightarrow \mathbb{P}^3$  with centre  $N$  lies on a conic  $\mathcal{Q}$  (which may be reducible), and the double cover of  $\mathcal{Q}$  ramified over this divisor is a stable curve  $\mathcal{X}$  of arithmetic genus 2 such that  $\mathcal{I}_{\mathcal{X}} \cong \mathcal{I}_{\mathcal{H}}/S$ .*

It is crucial to note that Theorem 8.1 is not constructive: it does not in itself yield the hyperplane  $H$ , nor the centre  $N$  of the projection to  $\mathbb{P}^3$ . It is noted in [7, §3.4] that  $H$  is defined by  $\phi^*(\Theta_{\mathcal{X}})$ , but in our situation we do not yet have an expression for  $\phi$  or  $\Theta_{\mathcal{X}}$ .

In the case  $\ell = 3$ , we are saved by a happy coincidence:  $2\ell - 1 = 5$ , so  $H = W$  (we return to this case in §11 below). For  $\ell > 3$ , we must compute  $H$  in some other way; Lemma 8.2, an easy corollary of Lemma 7.1, characterizes the possible hyperplanes.

**Lemma 8.2.** *The linear system of all hyperplanes in  $\mathbb{P}^{2\ell}$  containing  $W$  is generated by the  $2\ell - 5$  hyperplanes  $V(W_i)$  for  $0 \leq i \leq 2\ell - 6$ . That is, if  $H \supset W$  is a hyperplane in  $\mathbb{P}^{2\ell}$ , then*

$$H = V(\alpha_0 W_0 + \cdots + \alpha_{2\ell-6} W_{2\ell-6})$$

*for some  $(\alpha_0 : \cdots : \alpha_{2\ell-6})$  in  $\mathbb{P}^{2\ell-6}(\mathbb{k})$ .*

In view of Lemma 8.2, one naïve approach to computing  $H$  for  $\ell > 3$  would be to take a generic  $H = V(\sum_{i=0}^{2\ell-6} \alpha_i W_i)$  and compute its intersection with the secants  $\mathcal{L}_e$ . This yields  $(\ell^2 - 1)/2$  points whose coordinates are linear expressions in the  $\alpha_i$ . We could then solve for the values of  $\alpha_i$  by computing the zero locus of the  $(2\ell-2) \times (2\ell-2)$  minors of the matrix formed by the intersections  $H \cap \mathcal{L}_e$ ; but each minor is still a degree- $(2\ell-2)$  polynomial in  $2\ell-5$  variables, and the number of minors is exponential in  $\ell$ .

Alternatively, we could take a generic set of linear equations determining  $N$  inside the generic  $H$ ; requiring that this centre intersects any one of the  $(\ell^2 - 1)/2$  secants imposes  $O(\ell^4)$  quartic polynomial conditions on the  $O(\ell)$  unknowns.

In each approach the system is highly overdetermined, and with a clever choice of minors we might hope to get lucky and find solutions for toy examples. However, both approaches already represent a significant undertaking for  $\ell = 5$ , even over finite fields; they are totally impractical for larger  $\ell$  and for infinite fields.

We continue the treatment for general  $\ell$  in §9 and §10, supposing that an equation for  $H$  has been found; without such an equation, the `avIsogenies` package [1] represents a much more sensible approach for  $\ell \geq 5$  (if  $\mathbb{k}$  is finite). For  $\ell = 3$ , the Dolgachev–Lehavi method is as practical as it is interesting; we specialize to this case in §11 and §12.

## 9. FROM THEORY TO PRACTICE

To compute  $\mathcal{X}$  via Theorem 8.1, we must compute the map

$$\Phi := \pi \circ \rho_{2\ell} : \mathcal{H} \rightarrow \mathbb{P}^3,$$

where  $\pi : \mathbb{P}^{2\ell} \rightarrow \mathbb{P}^3$  is the projection with centre  $N$ . Suppose that we have an equation

$$H : \sum_i \alpha_i W_i = 0$$

for  $H$ . We can then apply Lemma 6.2 to compute the centre  $N = \langle \mathcal{L}_e \cap H : e \in S \setminus \{0\} \rangle$ . Since  $N \subset H$ , we may compute  $v_{0,0}, \dots, v_{0,2\ell}, v_{1,0}, \dots, v_{1,2\ell}, v_{2,0}, \dots, v_{2,2\ell}$  in  $\mathbb{k}$  such that

$$N = V\left(\sum_{i=0}^{2\ell} v_{0,i} U_i, \sum_{i=0}^{2\ell} v_{1,i} U_i, \sum_{i=0}^{2\ell} v_{2,i} U_i, \sum_{i=0}^{2\ell-6} \alpha_i W_i\right).$$

(This amounts to computing the kernel of the matrix whose rows are formed by the coordinates of the  $\mathcal{L}_e \cap H$ ; the choice of  $\sum_{i=0}^6 \alpha_i W_i$  for the fourth defining equation will be convenient later in the procedure.)

Fixing coordinates on  $\mathbb{P}^3$ , the projection  $\pi$  with centre  $N$  is defined by

$$\pi : (U_0 : \dots : U_{2\ell}) \longmapsto (V_0 : V_1 : V_2 : V_3) = \left(\sum_{i=0}^{2\ell} v_{0,i} U_i, \sum_{i=0}^{2\ell} v_{1,i} U_i, \sum_{i=0}^{2\ell} v_{2,i} U_i, \sum_{i=0}^{2\ell-6} \alpha_i W_i\right);$$

the composed map  $\Phi = \pi \circ \rho_{2\ell}$  is then

$$\Phi : (X : Y : Z) \longmapsto (V_0 : V_1 : V_2 : V_3) = (\Phi_0(X, Z) : \Phi_1(X, Z) : \Phi_2(X, Z) : \Phi_3(X, Z)),$$

where

$$\Phi_0 := \sum_{i=0}^{2\ell} v_{0,i} X^i Z^{2\ell-i}, \quad \Phi_1 := \sum_{i=0}^{2\ell} v_{1,i} X^i Z^{2\ell-i}, \quad \Phi_2 := \sum_{i=0}^{2\ell} v_{2,i} X^i Z^{2\ell-i},$$

and

$$\Phi_3 := \sum_{i=0}^{2\ell-6} \alpha_i X^i Z^{2\ell-6-i} F(X, Z).$$

The image of  $\Phi$  is a rational curve of degree  $2\ell$  in  $\mathbb{P}^3$ . It lies on the Kummer surface  $\mathcal{K}_{\mathcal{X}}$  of the unknown codomain Jacobian  $\mathcal{J}_{\mathcal{X}}$ , and is therefore the intersection of a quadric and a cubic hypersurface in  $\mathbb{P}^3$  (see [9, Chapter XIII]):

$$\Phi(\mathbb{P}^1) = \tilde{\mathcal{Q}} \cap \tilde{\mathcal{C}} \quad \text{where} \quad \tilde{\mathcal{Q}} = V(\tilde{Q}(V_0, V_1, V_2, V_3)) \quad \text{and} \quad \tilde{\mathcal{C}} = V(\tilde{C}(V_0, V_1, V_2, V_3))$$

for some forms  $\tilde{Q}$  and  $\tilde{C}$  of degree 2 and 3, respectively. The forms  $\tilde{Q}$  and  $\tilde{C}$  generate the elimination ideal

$$(\tilde{Q}, \tilde{C}) = (V_0 - \Phi_0, V_1 - \Phi_1, V_2 - \Phi_2, V_3 - \Phi_3) \cap \mathbb{k}[V_0, V_1, V_2, V_3];$$

note that  $\tilde{Q}$  is uniquely determined, and  $\tilde{C}$  is determined modulo  $(V_0Q, V_1Q, V_2Q, V_3Q)$ .

The Weierstrass points of  $\mathcal{H}$  map into the hyperplane  $V_3 = 0$ , which we identify with  $\mathbb{P}^2$ . (This simplification motivates our choice of  $\Phi_3$ .) Theorem 8.1 asserts that a conic  $\mathcal{Q}$  passes through the six images, and indeed

$$\mathcal{Q} = V(Q(V_0, V_1, V_2)) \subset \mathbb{P}^2, \quad \text{where } Q(V_0, V_1, V_2) = \tilde{Q}(V_0, V_1, V_2, 0).$$

The image of the Weierstrass divisor under  $\Phi$  is therefore  $\mathcal{Q} \cap \mathcal{C}$ , where

$$\mathcal{C} = V(C(V_0, V_1, V_2)) \subset \mathbb{P}^2 \quad \text{with } C(V_0, V_1, V_2) = \tilde{C}(V_0, V_1, V_2, 0).$$

We are more interested in the forms  $Q$  and  $C$  than in  $\tilde{Q}$  and  $\tilde{C}$ , and it is a simple matter to interpolate them. For  $Q$ , we compute the six quintic polynomials  $\Phi_i\Phi_j(x, 1) \bmod F(x, 1)$  for  $0 \leq i \leq j \leq 2$ ; the unique linear relation between them (and between the  $v_{i,0}v_{j,0}$  if  $F_6 = 0$ ) yields the coefficients of  $Q$ . Similarly, to find  $C$  we compute the ten quintics  $\Phi_i\Phi_j\Phi_k(x, 1) \bmod F(x, 1)$  for  $0 \leq i \leq j \leq k \leq 2$ ; any one of the linear relations between them (and the  $v_{i,0}v_{j,0}v_{k,0}$  if  $F_6 = 0$ ) gives an equation for a valid cubic  $C$ .

## 10. THE CODOMAIN CURVE

The data  $(\mathcal{Q}, \mathcal{Q} \cap \mathcal{C})$  specifies a genus 2 curve  $\mathcal{X}$  (up to a quadratic twist) as a double cover of  $\mathcal{Q}$  ramified over the six points of  $\mathcal{Q} \cap \mathcal{C}$ . This is the output of the Dolgachev–Lehavi algorithm and of Theorem 8.1, and it is sufficient for computing isomorphism invariants of  $\mathcal{X}$  (see, for example, [4] and [12]).

In some situations, however, we would like to derive a defining equation for  $\mathcal{X}$  itself. When  $\mathcal{Q}$  is nonsingular, we recover a hyperelliptic curve; in the degenerate case where  $\mathcal{Q}$  is singular, we recover a union of two elliptic curves  $\mathcal{X}_+$  and  $\mathcal{X}_-$ , which are generally defined over a quadratic extension of  $\mathbb{k}$  (in which case they are Galois conjugates). The procedure is essentially standard (cf. [4, §2]), but we recall it here for completeness.

**Algorithm 10.1.** Computes a (possibly reducible) genus 2 curve representing a double cover of a given plane conic ramified over the intersection with a plane cubic.

**Input:** A plane conic  $\mathcal{Q} : Q(V_0, V_1, V_2) = 0$  and cubic  $\mathcal{C} : C(V_0, V_1, V_2) = 0$ .

**Output:** A genus 2 curve  $\mathcal{X}$  forming a double cover of  $\mathcal{Q}$  ramified over  $\mathcal{Q} \cap \mathcal{C}$ . If  $\mathcal{Q}$  is singular, then  $\mathcal{X}$  will be a one-point union of elliptic curves  $\mathcal{X}_+$  and  $\mathcal{X}_-$ , with  $\mathcal{X}_\pm$  ramified over  $P_0$  and  $\mathcal{C} \cap \mathcal{L}_\pm$  where  $\mathcal{Q} = \mathcal{L}_+ + \mathcal{L}_-$  and  $P_0 = \mathcal{L}_+ \cap \mathcal{L}_-$ .

**1:** Let  $M$  be the matrix defined by

$$M := \begin{pmatrix} 2q_{0,0} & q_{0,1} & q_{0,2} \\ q_{0,1} & 2q_{1,1} & q_{1,2} \\ q_{0,2} & q_{1,2} & 2q_{2,2} \end{pmatrix}, \quad \text{where } \sum_{0 \leq i \leq j \leq 2} q_{i,j} V_i V_j = Q(V_0, V_1, V_2).$$

**2:** If  $\det(M) = 0$ , then  $\mathcal{Q}$  is singular.

**2a:** Compute a diagonal matrix  $D = \text{diag}(a, b, 0)$  and an invertible matrix  $T$  such that  $M = TDT^{-1}$ .

**2b:** Set  $\delta = \sqrt{-a/b}$ , and define homogeneous cubics  $C_+(X, Z)$  and  $C_-(X, Z)$  by  $C_\pm := C((t_{00} \pm \delta t_{01})Z + t_{02}X, (t_{10} \pm \delta t_{11})Z + t_{12}X, (t_{20} \pm \delta t_{21})Z + t_{22}X)$  where

$$\begin{pmatrix} t_{00} & t_{01} & t_{02} \\ t_{10} & t_{11} & t_{12} \\ t_{20} & t_{21} & t_{22} \end{pmatrix} = T.$$



**2c:** Define elliptic curves  $\mathcal{X}_+$  and  $\mathcal{X}_-$  over  $\mathbb{k}(\delta)$  in  $\mathbb{P}(2, 3, 2)$  by

$$\mathcal{X}_+ : Y^2 = C_+(X, Z) \quad \text{and} \quad \mathcal{X}_- : Y^2 = C_-(X, Z),$$

and return the union of  $\mathcal{X}_+$  and  $\mathcal{X}_-$  identifying the points at infinity.

**3:** Otherwise,  $\mathcal{Q}$  is nonsingular.

**3a:** Compute a rational point  $P = (\alpha_0 : \alpha_1 : \alpha_2)$  in  $\mathcal{Q}(\mathbb{k})$  (see Remark 10.2).

**3b:** Let  $\pi : \mathbb{P}^1 \rightarrow \mathcal{Q}$  be the corresponding parametrization, defined by

$$\pi : (X : Z) \mapsto (V_0 : V_1 : V_2) = (P_0(X, Z) : P_1(X, Z) : P_2(X, Z))$$

(the  $P_i$  are quadratic forms).

**3c:** Return  $\mathcal{X} : Y^2 = C(P_0(X, Z), P_1(X, Z), P_2(X, Z))$ .

*Remark 10.2.* Step 3a of Algorithm 10.1 requires us to compute a  $\mathbb{k}$ -rational point  $P$  on the conic  $\mathcal{Q}$ . If  $\mathcal{H}$  has a rational Weierstrass point  $W_0$ , then we may take  $P = \Psi(W_0)$ . Generically, however,  $\mathcal{H}$  has no rational Weierstrass points, and then we are obliged to search for a rational point on  $\mathcal{Q}$ . We are guaranteed that such a rational point exists (cf. [12, Lemme 1]). Over a finite field, finding a rational point is straightforward; over the rationals, we can apply (for example) the Cremona–Rusin algorithm [6].

## 11. THE ALGORITHM FOR $\ell = 3$

Consider now the special case  $\ell = 3$ . The map  $\rho_6 : \mathcal{H} \rightarrow \mathcal{R}_6 \subset \mathbb{P}^6$  is defined by

$$\rho_6 : (X : Y : Z) \mapsto (U_0 : U_1 : \cdots : U_5 : U_6) = (Z^6 : XZ^5 : \cdots : X^5Z : X^6).$$

The hyperplane  $H$  of Theorem 8.1 contains  $W = \langle \rho_6(W_{\mathcal{H}}) \rangle$  by definition; but  $\dim H = \dim W = 5$ , so  $H = W$ . Applying Lemma 7.1, we find

$$(3) \quad H = V(W_0) = V\left(\sum_{i=0}^6 F_i U_i\right) \subset \mathbb{P}^6;$$

this allows us to simplify Lemma 6.2 for the case  $\ell = 3$ .

**Proposition 11.1.** *If  $e = \langle a, b, d \rangle$  is a nonzero 3-torsion point of  $\mathcal{F}_{\mathcal{H}}$ , then*

$$H \cap \mathcal{L}_e = (\gamma_0(e) : \cdots : \gamma_6(e)),$$

where the  $\gamma_i$  are defined as follows:

- (1) If  $a = 1$ , then  $\gamma_i(e) = 0$  for  $0 \leq i < 5$ , with  $\gamma_5(e) = F_6$  and  $\gamma_6(e) = -F_5$ .
- (2) If  $a$  is linear, then  $\gamma_i(e) = 0$  for  $0 \leq i < 6$ , and  $\gamma_6(e) = 1$ .
- (3) If  $a(x) = x^2 + a_1x + a_0$  with  $a_1^2 \neq 4a_0$ , then

$$\gamma_i(e) = \sum_{j=0}^6 F_j (a_2^j \sigma_{i-j} + a_2^i \sigma_{j-i}) \quad \text{for } 0 \leq i \leq 6$$

with  $\sigma_k = 0$  for  $k < 1$ ,  $\sigma_1 = 1$ , and  $\sigma_k = -a_1 \sigma_{k-1} - a_2 \sigma_{k-2}$  for  $k > 1$ .

- (4) If  $a(x) = x^2 + a_1x + a_0$  with  $a_1^2 = 4a_0$ , then

$$\gamma_i(e) = \sum_{j=0}^6 (i-j) F_j (-a_1/2)^{i+j-1} \quad \text{for } 0 \leq i \leq 6.$$

*Proof.* This follows immediately from Lemma 6.2 on setting  $H = V(\sum_{i=0}^6 F_i U_i)$  and noting that  $a(x)$  cannot divide  $h(x) = \sum_{i=0}^6 F_i x^i$  (since otherwise  $e$  would have order 2).  $\square$

We now present a version of the Dolgachev–Lehavi algorithm for  $\ell = 3$  based on the extended Mumford representation. The algorithm requires only elementary matrix algebra and polynomial arithmetic, and should be easily implemented in most computational algebra systems.

**Algorithm 11.2.** A streamlined Dolgachev–Lehavi-style algorithm for  $\ell = 3$ .

**Input:** A genus 2 curve  $\mathcal{H} : Y^2 = F(X, Z) = \sum_{i=0}^6 F_i X^i Z^{6-i}$  over  $\mathbb{k}$  and a maximal Weil-isotropic subgroup  $S$  of  $\mathcal{I}_{\mathcal{H}}[3]$ , its elements defined over  $\mathbb{k}$  and presented as in §4.

**Output:** A genus 2 curve  $\mathcal{X} / \mathbb{k}$  such that there exists an isogeny  $\phi : \mathcal{I}_{\mathcal{H}} \rightarrow \mathcal{I}_{\mathcal{X}}$  with kernel  $S$  (the curve  $\mathcal{X}$  is computed up to a quadratic twist, so the isogeny may only be defined over a quadratic extension of  $\mathbb{k}$ ).

- 1: Compute a minimal subset  $S^\pm$  of  $S$  such that  $S = \{e : e \in S^\pm\} \cup \{-e : e \in S^\pm\} \cup \{0\}$  (then  $\{\mathcal{L}_e : e \in S^\pm\} = \{\mathcal{L}_e : e \in S \setminus \{0\}\}$ ; this avoids redundancy in Steps 2 and 3).
- 2: For each  $e$  in  $S^\pm$ , compute the vector  $v_e = (\gamma_0(e), \dots, \gamma_6(e))$  using the formulæ in Proposition 11.1.
- 3: Compute vectors  $n_i = (v_{i,0}, \dots, v_{i,6})$  such that  $\{n_0, n_1, n_2, (F_j : 0 \leq j \leq 6)\}$  is a basis for the (left) kernel of the  $7 \times 4$  matrix  $(v_e^t : e \in S^\pm)$ . Set

$$\Phi_i = \sum_{j=0}^6 v_{i,j} X^j Z^{6-j} \quad \text{for } 0 \leq i \leq 2.$$

- 4: For each  $0 \leq i \leq j \leq 2$ , compute the vector  $r_{i,j}$  of length 6 whose  $n^{\text{th}}$  entry is the coefficient of  $x^{n-1}$  in  $(\Phi_i \Phi_j)(x, 1) \bmod F(x, 1)$ . If  $F_6 = 0$ , then take the 6<sup>th</sup> entry of  $r_{i,j}$  to be  $v_{i,0} v_{j,0}$ : this allows us to correctly interpolate through the image of the Weierstrass point at infinity.
- 5: Compute a generator  $(q_{i,j} : 0 \leq i \leq j \leq 2)$  for the (left) kernel of the  $6 \times 6$  matrix whose rows are the  $r_{i,j}$  for  $0 \leq i \leq j \leq 2$ . Set

$$Q(V_0, V_1, V_2) := q_{0,0} V_0^2 + q_{0,1} V_0 V_1 + q_{0,2} V_0 V_2 + q_{1,1} V_1^2 + q_{1,2} V_1 V_2 + q_{2,2} V_2^2.$$

- 6: For each  $0 \leq i \leq j \leq k \leq 2$ , compute the vector  $s_{i,j,k}$  of length 6 whose  $n^{\text{th}}$  entry is the coefficient of  $x^{n-1}$  in  $(\Phi_i \Phi_j \Phi_k)(x, 1) \bmod F(x, 1)$ . If  $F_6 = 0$ , then take the 6<sup>th</sup> entry of  $s_{i,j,k}$  to be  $v_{i,0} v_{j,0} v_{k,0}$ .
- 7: Compute any nontrivial element  $(c_{i,j,k} : 0 \leq i \leq j \leq k \leq 2)$  of the (left) kernel of the  $10 \times 6$  matrix whose rows are the  $s_{i,j,k}$  for  $0 \leq i \leq j \leq k \leq 2$ , and set

$$C(V_0, V_1, V_2) := \sum_{0 \leq i \leq j \leq k \leq 2} c_{i,j,k} V_i V_j V_k.$$

- 8: Return the result  $\mathcal{X}$  of Algorithm 10.1 applied to  $\mathcal{Q} = V(Q)$  and  $\mathcal{C} = V(C)$ .

## 12. THE ALGORITHM IN PRACTICE

We conclude with an example for  $\ell = 3$ . To avoid a visually overwhelming mass of coefficients, we will work over a small finite field; the curve was chosen at random.

Consider the genus 2 curve over  $\mathbb{F}_{997}$  defined by

$$\mathcal{H} : Y^2 = X^6 + 113X^5Z + 99X^4Z^2 + 363X^3Z^3 + 64X^2Z^4 + 503XZ^5 + 630Z^6.$$

Computing the zeta function of  $\mathcal{H}$  (using Magma), we see that its Weil polynomial is

$$P(T) = T^4 - 31T^3 + 54T^2 - 30907T + 994009,$$

so  $\mathcal{I}_{\mathcal{H}}$  is absolutely simple by the Howe–Zhu criterion [8, Theorem 6]. The elements  $D_1 = \langle x^2 + 392x + 208, 579x + 603, 2 \rangle$  and  $D_2 = \langle x^2 + 48x + 527, 918x + 832, 2 \rangle$  of  $\mathcal{I}_{\mathcal{H}}$  have order 3, and  $S = \langle D_1, D_2 \rangle$  is a maximal 3-Weil isotropic subgroup of  $\mathcal{I}_{\mathcal{H}}[3]$ . Applying Algorithm 11.2, we may take

$$S^\pm = \left\{ \begin{array}{l} \langle x^2 + 392x + 208, 579x + 603, 2 \rangle, \langle x^2 + 48x + 527, 918x + 832, 2 \rangle, \\ \langle x^2 + 428x + 880, 252x + 901, 2 \rangle, \langle x^2 + 348x + 292, 596x + 269, 2 \rangle \end{array} \right\}$$

in Step 1. Equation (3) shows that the hyperplane  $H \subset \mathbb{P}^6$  is defined by

$$H : 630U_0 + 503U_1 + 64U_2 + 363U_3 + 99U_4 + 113U_5 + U_6 = 0,$$

so the matrix in Step 3 is

$$\begin{pmatrix} 234 & 319 & 906 & 896 \\ 780 & 16 & 29 & 754 \\ 500 & 565 & 703 & 398 \\ 680 & 329 & 823 & 248 \\ 324 & 68 & 779 & 868 \\ 742 & 416 & 468 & 392 \\ 664 & 395 & 698 & 952 \end{pmatrix};$$

computing kernel vectors, we take

$$\begin{aligned} \Phi_0 &= 121X^6 + 742X^5Z + 549X^4Z^2 + XZ^5, \\ \Phi_1 &= 285X^6 + 642X^5Z + 332X^4Z^2 + X^2Z^4, \\ \Phi_2 &= 889X^6 + 701X^5Z + 454X^4Z^2 + X^3Z^3. \end{aligned}$$

The quadratic form of Step 5 is then

$$Q(V_0, V_1, V_2) = V_0^2 + 52V_0V_1 + 361V_1^2 + 548V_0V_2 + 715V_1V_2 + 296V_2^2,$$

and we may take the cubic form in Step 7 to be

$$C(V_0, V_1, V_2) = V_0^3 + 167V_1^3 + 149V_0V_1V_2 + 836V_1^2V_2 + 885V_0V_2^2 + 538V_1V_2^2 + 294V_2^3.$$

We now apply Algorithm 10.1 to  $\mathcal{Q} : Q(V_0, V_1, V_2) = 0$  and  $\mathcal{C} : C(V_0, V_1, V_2) = 0$ . The conic  $\mathcal{Q}$  is nonsingular, and  $\mathcal{C}$  has a rational Weierstrass point  $(-76 : 0 : 1)$  mapping to the point  $(-36 : -80 : 1)$  on  $\mathcal{Q}$ . The associated parametrization  $\mathbb{P}^1 \rightarrow \mathcal{Q}$  is defined by

$$(X : Z) \longmapsto (36X^2 + 781XZ + 109Z^2 : 80X^2 + 865XZ + 17Z^2 : 996X^2 + 945XZ + 636Z^2);$$

substituting its defining polynomials into  $C$ , we find that  $\mathcal{X}$  has a model

$$\mathcal{X} : Y^2 = 118X^5Z + 183X^4Z^2 + 613X^3Z^3 + 35X^2Z^4 + 174XZ^5 + 474Z^6.$$

In fact, this is the quadratic twist of the true  $\mathcal{X}$ : explicit calculation shows that its Weil polynomial is  $P(-T)$ .

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