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## A model-free no-arbitrage price bound for variance options\*

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**Abstract:** In the framework of Galichon, Henry-Labordère and Touzi [9], we consider the model-free no-arbitrage bound of variance option given the marginal distributions of the underlying asset. We first make some approximations which restrict the computation on a bounded domain. Then we propose a gradient projection algorithm together with a finite difference scheme to approximate the bound. The general convergence result is obtained. We also provide a numerical example on the *variance swap* option.

**Key-words:** Variance option, model-free price bound, gradient projection algorithm.

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## Un borne de valeur sans arbitrage, indépendante d'un modèle, d'options sur variance

**Résumé :** Dans le cadre de Galichon, Henry-Labordère et Touzi [9], nous considérons la borne sans arbitrage, indépendante d'un modèle, étant donné la distribution marginale du sous-jacent. Nous restreignons d'abord le calcul à un domaine borné. Puis nous proposons un algorithme de gradient avec projection, combiné à un schéma de différences finies, pour approcher la borne. Nous obtenons un résultat général de convergence, puis traitons un exemple numérique d'option sur swap.

**Mots-clés :** Option sur variance, borne de prix indépendante d'un modèle, algorithme de gradient avec projection.

## 1 Introduction

In a recent work of Galichon, Henry-Labordère and Touzi [9], the authors proposed a framework to compute the optimal model-free no-arbitrage price bound of exotic options in a vanilla-liquid market. Let  $\Omega^d := C([0, T], \mathbb{R}^d)$  be the canonical space with canonical process  $X$  and canonical filtration  $\mathbb{F}^d = (\mathcal{F}_t^d)_{0 \leq t \leq T}$ ,  $S_0$  be a constant. We denote by  $\mathcal{P}(\delta_{S_0})$  the collection of all probability measures  $\mathbb{P}$  on  $(\Omega^d, \mathcal{F}_T^d)$  under which  $X$  is a  $\mathbb{F}^d$ -martingale and  $X_0 = S_0$   $\mathbb{P}$ -a.s. As indicated in [9], there is a progressively measurable process  $\langle X \rangle_t$  which is pathwise defined and coincides with the  $\mathbb{P}$ -quadratic variation of  $X$ ,  $\mathbb{P}$ -a.s. for every  $\mathbb{P} \in \mathcal{P}(\delta_{S_0})$ .

The process  $X$  is a candidate of underlying stock price, we do not impose any dynamic assumptions on  $X$ , but only suppose that it is a martingale. Then for an option with payoff  $G \in \mathcal{F}_T^d$ , the upper bound of model-free no-arbitrage price is given by

$$\sup_{\mathbb{P} \in \mathcal{P}(\delta_{S_0})} \mathbb{E}^{\mathbb{P}}[G].$$

Suppose in addition that we are in a market where the vanilla options with maturity  $T$  are liquid, so that the investor can identify the marginal distribution  $\mu$  of  $X_T$ . In other words, let  $\phi \in \mathbb{L}^1(\mathbb{R}^d, \mu)$ , the  $T$ -maturity European option with payoff  $\phi(X_T)$  has a unique no-arbitrage price

$$\mu(\phi) := \int_{\mathbb{R}^d} \phi(x) \mu(dx).$$

Let us use the vanilla option portfolio to hedge  $G$ . By buying a portfolio  $\phi(X_T)$ , we spend  $\mu(\phi)$  and so the payoff at maturity  $T$  becomes  $G - \phi(X_T)$ . Therefore, we get a new upper bound of model-free price:  $\sup_{\mathbb{P} \in \mathcal{P}(\delta_{S_0})} \mathbb{E}^{\mathbb{P}}[G - \phi(X_T)] + \mu(\phi)$ . By minimizing on the vanilla option portfolio  $\phi$ , the optimal upper bound is then given by

$$\inf_{\phi \in \mathbb{L}^1(\mu)} \sup_{\mathbb{P} \in \mathcal{P}(\delta_{S_0})} \{ \mathbb{E}^{\mathbb{P}}[G - \phi(X_T)] + \mu(\phi) \}. \quad (1.1)$$

As another motivation, we observe that the upper bound (1.1) is formally the conjugate dual formulation of problem

$$\sup_{\mathbb{P} \in \mathcal{P}(\delta_{S_0}, \mu)} \mathbb{E}^{\mathbb{P}}[G] = \sup_{\mathbb{P} \in \mathcal{P}(\delta_{S_0})} \inf_{\phi \in \mathbb{L}^1(\mu)} \{ \mathbb{E}^{\mathbb{P}}[G - \phi(X_T)] + \mu(\phi) \}, \quad (1.2)$$

where  $\mathcal{P}(\delta_{S_0}, \mu)$  denotes the collection of all martingale probability measures  $\mathbb{P} \in \mathcal{P}(\delta_{S_0})$  such that  $X_T \sim^{\mathbb{P}} \mu$ . We remark that the above equality holds since

$$\inf_{\phi \in \mathbb{L}^1(\mu)} \{ \mathbb{E}^{\mathbb{P}}[G - \phi(X_T)] + \mu(\phi) \} = \begin{cases} \mathbb{E}^{\mathbb{P}}[G] & \text{if } X_T \sim^{\mathbb{P}} \mu, \\ -\infty & \text{otherwise.} \end{cases}$$

In this paper, we shall consider in particular the no-arbitrage price bound of variance option in a similar framework. Let us restrict to the one-dimensional case  $d = 1$  and  $T_1 > T_0 \geq 0$  be two constants. We define the corresponding canonical space as  $\Omega := C([0, T_1], \mathbb{R})$  and denote still by  $X$  the canonical process,  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T_1}$  the canonical filtration and by  $\langle X \rangle$  the progressively measurable process which coincides with the quadratic variation of  $X$  under every martingale probability measure  $\mathbb{P}$ . Suppose that the vanilla options of maturities  $T_0, T_1$  are liquid such that we can identify the marginal distribution  $\mu_0$  (resp.  $\mu_1$ ) for  $X_{T_0}$  (resp.  $X_{T_1}$ ). We shall consider the variance option with payoff

$$G := g(\langle X \rangle_{T_0, T_1}, X_{T_1}) \text{ at maturity } T_1 \text{ for some appropriate function } g,$$

where  $\langle X \rangle_{T_0, T_1} := \langle X \rangle_{T_1} - \langle X \rangle_{T_0}$ . Let  $\mathcal{P}^2(\mu_0)$  denotes the set of all the probability measures  $\mathbb{P}$  on  $(\Omega, \mathcal{F}_{T_1})$  such that  $X_{T_0} \sim^{\mathbb{P}} \mu_0$  and  $\mathbb{E}^{\mathbb{P}}[\langle X \rangle_{T_0, T_1} | \mathcal{F}_{T_0}] < \infty$ ,  $\mathbb{P}$ -a.s., we define the no-arbitrage price upper bound of variance option  $G = g(\langle X \rangle_{T_0, T_1}, X_{T_1})$  by

$$\inf_{\phi \in \text{Quad}} \sup_{\mathbb{P} \in \mathcal{P}^2(\mu_0)} \left\{ \mathbb{E}^{\mathbb{P}} [g(\langle X \rangle_{T_0, T_1}, X_{T_1}) - \phi(X_{T_1})] + \mu_1(\phi) \right\}, \quad (1.3)$$

where Quad denotes the set of functions satisfying a quadratic growth condition, i.e.

$$\text{Quad} := \left\{ \phi : \mathbb{R} \rightarrow \mathbb{R} \text{ such that } \sup_{x \in \mathbb{R}} \frac{|\phi(x)|}{1 + |x|^2} < \infty \right\}. \quad (1.4)$$

**Remark 1.1.** *The main reason to choose Quad is from the observation of Dupire [7] that variance swap is equivalent to a European option with payoff  $X_T^2$ , see also Remark 2.3 and Corollary 3.9.*

By the time-change martingale theorem (see e.g. Theorem 3.4.6 of Karatzas and Shreve [12]), we can establish a correspondence between the set of martingale probability measures on  $(\Omega, \mathcal{F}_{T_1})$  and the set of stopping times on a Brownian motion. In fact, a local martingale  $Y$  can be represented as a time-changed Brownian motion, i.e.  $Y_t = W_{\langle Y \rangle_t}$  with a Brownian motion  $W$ . On the other hand, given a stopping time  $\tau$  on  $W$ , the process  $Y$  defined by  $Y_t := W_{\tau \wedge \frac{t}{T-t}}$  is a local martingale between 0 and  $T$ . Therefore, (1.3) can be formulated as

$$\bar{U} := \inf_{\phi \in \text{Quad}} \bar{u}(\phi) \quad \text{with} \quad \bar{u}(\varphi) := \sup_{\tau \in \mathcal{T}} \mathbb{E}[g(\tau, W_\tau) - \phi(W_\tau)] + \mu_1(\phi), \quad (1.5)$$

where  $W$  is a Brownian motion such that  $W_0 \sim \mu_0$  and

$$\mathcal{T} := \left\{ \tau \text{ stopping times such that } \mathbb{E}[\tau | W_0] < \infty, \text{ a.s.} \right\}. \quad (1.6)$$

We can also derive a dual formulation for (1.5) following the same arguments as for deriving (1.2). Let  $\mathcal{T}(\mu_1)$  denote the set of all stopping times  $\tau \in \mathcal{T}$  such that  $W_\tau \sim \mu_1$ , then the dual formulation of (1.5) becomes

$$\sup_{\tau \in \mathcal{T}} \inf_{\phi \in \text{Quad}} \mathbb{E}[g(\tau, W_\tau) - \phi(W_\tau)] + \mu_1(\phi) = \sup_{\tau \in \mathcal{T}(\mu_1)} \mathbb{E}[g(\tau, W_\tau)]. \quad (1.7)$$

Given a Brownian motion  $W$  and a distribution  $\mu_1$ , the problem of finding stopping time  $\tau$  such that  $W_\tau \sim \mu_1$ , i.e.  $\tau \in \mathcal{T}(\mu_1)$ , is called the Skorokhod Embedding Problem (SEP). Then our formulation (1.5) is consistent with Hobson's [10] observation of the connection between the SEP and the problem of optimal no-arbitrage bounds of exotic options in a vanilla-liquid market.

The SEP and the optimality property of its solutions as well as their applications in finance are studied in several papers recently, we refer to Obłój [15] and Hobson [11] for a survey. In particular, for the optimization problem (1.7), if  $g(x, t) = f(t)$  for some function  $f$  defined on  $\mathbb{R}^+$ , it is proved that the maximum is achieved by Root's embedding when  $f$  is concave and by Röst's embedding when  $f$  is convex (see Root [16] and Rost [17]). However, for general payoff function  $g$ , there is no systematic method to find the optimal value of such problems. That is also our main motivation to develop a numerical method to solve these problems.

Our main contribution is then to provide a numerical scheme to approximate the bounds for general variance options.

The rest of the paper is organized as follows: In Section 2, we give an equivalent formulation for the bound  $\bar{U}$  in (1.5). Then in Section 3 we provide an asymptotic analysis of our approximation,

which restrict the calculation of  $\bar{U}$  to a bounded domain. In Section 4, we propose a numerical scheme which combines the gradient projection algorithm and the finite difference method, and we give a general convergence result. Finally, Section 5 provides a numerical example on *variance swap*.

**Notations:** Let  $\mu$  be a probability measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , we define

$$\mu(\phi) := \int_{\mathbb{R}} \phi(x) \mu(dx), \quad \text{for every } \phi \in \mathbb{L}^1(\mu).$$

## 2 An equivalent formulation of the bound

We will fix the payoff function  $g : (t, x) \in \mathbb{R}^+ \times \mathbb{R} \mapsto g(t, x) \in \mathbb{R}$  of the variance option as well as the marginal distributions  $\mu_0, \mu_1$ , and then reformulate the price bound problem (1.5). To make the problem be well posed, let us first make some assumptions on the marginal distributions  $\mu_0, \mu_1$  and the payoff function  $g$ .

**Assumption 1.** *The probability measures  $\mu_0, \mu_1$  on  $\mathbb{R}$  have finite second moment, i.e.*

$$\mu_0(\phi_0) + \mu_1(\phi_0) < \infty, \quad \text{with } \phi_0(x) := x^2.$$

Moreover,  $\mu_0 \leq \mu_1$  in the convex order, i.e.

$$\mu_0(\phi) \leq \mu_1(\phi), \quad \text{for every convex function } \phi \text{ defined on } \mathbb{R}. \quad (2.1)$$

**Remark 2.1.** *It is shown in Strassen [18] that the convex order inequality (2.1) is a sufficient and necessary condition for the existence of a martingale with marginal distributions  $\mu_0$  and  $\mu_1$  at time  $T_0$  and  $T_1$  such that  $T_0 < T_1$ .*

*In particular, since the identity function  $I$  (where  $I(x) := x$ ) and its opposite  $-I$  are both convex, it follows immediately from (2.1) that  $\mu_0$  and  $\mu_1$  have the same first moment, i.e.  $\mu_0(I) = \mu_1(I)$ .*

**Assumption 2.** *The payoff function  $g(t, x)$  is  $L_0$ -Lipschitz in  $(t, x)$  with constant  $L_0 \in \mathbb{R}^+$ .*

**Example 2.2.** *The most popular variance option is the “variance swap”, whose payoff function is  $g(t, x) = t$ . There exist also “volatility swap” with payoff  $g(t, x) = \sqrt{t}$ , and calls (puts) on variance, or volatility, where the payoff function are  $(t - K)^+$  ( $(K - t)^+$ ), or  $(\sqrt{t} - K)^+$  ( $(K - \sqrt{t})^+$ ).*

In addition to Assumption 2, we give another assumption on the payoff function  $g$ .

**Assumption 3.** *The function  $g(t, x)$  increases in  $t$ , and convex in  $x$  for every fixed  $t \in \mathbb{R}^+$ . Moreover, for every fixed  $t \in \mathbb{R}^+$ ,  $g(t, 0) = \min_{x \in \mathbb{R}} g(t, x)$  and  $g(t, x)$  is affine in  $x$  on  $[M_0, \infty)$  and  $(-\infty, -M_0]$  with constant  $M_0 \in \mathbb{R}^+$ .*

**Remark 2.3.** *Assumption 3 may not be crucial given Assumptions 1 and 2. As we shall see later in Corollary 3.9, let  $K \in \mathbb{R}$  and  $\psi$  be defined on  $\mathbb{R}$ , denote  $g_{K, \psi}(t, x) := g(t, x) + Kt + \psi(x)$ , we then have*

$$\bar{U}(g_{K, \psi}) = \bar{U}(g) + KC_0 + \mu_1(\psi),$$

where  $\bar{U}(g)$  (resp.  $\bar{U}(g_{K, \psi})$ ) denotes the upper bound of (1.5) associated with the payoff function  $g$  (resp.  $g_{K, \psi}$ ), and

$$C_0 := \mu_1(\phi_0) - \mu_0(\phi_0), \quad \text{with } \phi_0(x) := x^2. \quad (2.2)$$

Therefore, for an arbitrary payoff function  $g$ , we can consider the payoff function  $g(t, x) + Kt$  which is increasing in  $t$ . And this does not change the nature of the upper bound problem (1.5).



Now we shall give an equivalent formulation of the problem (1.5). Let  $B = (B_t)_{t \geq 0}$  be a standard one-dimensional Brownian motion such that  $B_0 = 0$ ,  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  be its natural filtration and  $\mathcal{T}^\infty$  be a set of  $\mathbb{F}$ -stopping times defined by

$$\mathcal{T}^\infty := \{ \mathbb{F}\text{-stopping time } \tau \text{ such that } \mathbb{E}(\tau) < \infty \}. \quad (2.3)$$

Given a strategy function  $\phi \in \text{Quad}$  which is given by (1.4), we denote

$$g^\phi(t, x) := g(t, x) - \phi(x), \quad (2.4)$$

and define functions  $\lambda^\phi : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\lambda_0^\phi : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\lambda^\phi(t, x) := \sup_{\tau \in \mathcal{T}^\infty} \mathbb{E} [ g^\phi(t + \tau, x + B_\tau) ], \quad \text{and } \lambda_0^\phi(\cdot) := \lambda^\phi(0, \cdot). \quad (2.5)$$

Then the new formulation of the model-free no-arbitrage price upper bound is given by

$$U := \inf_{\phi \in \text{Quad}} u(\phi), \quad \text{with } u(\phi) := \mu_0(\lambda_0^\phi) + \mu_1(\phi). \quad (2.6)$$

We notice that  $\mu_0(\lambda_0^\phi)$  is well defined under Assumptions 1 and 2, by the fact that  $\lambda_0^\phi(x) \geq g^\phi(0, x) = g(0, x) - \phi(x) \geq -C(1 + x^2)$  for some positive constant  $C$  and that  $\lambda^\phi(t, x)$  is measurable from the following Lemma.

**Lemma 2.4.** *Let Assumptions 1 and 2 hold, then for every  $\phi \in \text{Quad}$ , the function  $\lambda^\phi(t, x)$  is lower-semicontinuous and hence measurable.*

**Proof.** By Assumption 2, for a fixed  $\phi \in \text{Quad}$ , there is a constant  $C \in \mathbb{R}^+$  such that

$$| g^\phi(t + \tau, x + B_\tau) | \leq C(1 + t + \tau + x^2 + B_\tau^2).$$

Thus for a fixed  $\tau \in \mathcal{T}^\infty$ ,  $(t, x) \mapsto \mathbb{E}[g^\phi(t + \tau, x + B_\tau)]$  is continuous by the dominated convergence theorem together with (3.14) proved below. It follows immediately by its definition in (2.5) that  $\lambda^\phi$  is lower-semicontinuous since it is represented as the supremum of a family of continuous function.  $\square$

**Theorem 2.5.** *Let Assumptions 1, 2 and 3 hold. Then the problem (1.5) and (2.6) are equivalent, i.e.  $\bar{U} = U$ .*

The proof is a simple consequence of the dynamic programming, we shall report it in Appendix.

**Remark 2.6.** *Here we only give the upper bound formulation. By the symmetry of the set Quad defined in (1.4), if we reverse the payoff function to  $-g(t, x)$ , then with the upper bound  $U(-g)$  associated to payoff  $-g$ , the value  $-U(-g)$  is the lower bound for the payoff  $g$ .*

When  $g(t, x) = (t - K)^+$ , i.e. the option is the variance call, Dupire [7], Carr and Lee [6] proposed a systematic scheme to find a non-optimal bound as well as the associated strategy  $\phi$  in a similar context. In their implemented examples, they showed that their bounds are quite close to the optimal bounds from Root's embedding solution.

For general payoff functions  $g(t, x)$ , when there is no systematic method to solve the problem (2.6), we shall propose a numerical scheme to approximate the optimal  $\phi$  as well as the optimal upper bound  $U$ . In fact, we can easily observe that  $\phi \mapsto \lambda^\phi$  is convex since it is represented as the supremum of a family of linear mapping in (2.5). Thus  $\phi \mapsto u(\phi)$  is a convex function and the problem of  $U$  in (2.6) turns out to be a minimization problem of a convex function, as expected for a dual formulation of (1.7). We propose to use the finite difference scheme to solve  $u(\phi)$  with every given  $\phi$ , and then approximate the minimization problem on  $\phi$  by an iterative algorithm.

### 3 Analytic approximation

In order to make the numerical resolution of  $U$  in (2.6) possible, we shall first restrict the calculations to a bounded domain by some analytic approximations.

#### 3.1 The analytic approximation in four steps

Let us present the analytic approximation in four steps. The first step is to introduce a subset of  $\text{Quad}$  defined by

$$\text{Quad}_0 := \{ \phi \in \text{Quad} \text{ non negative, convex, such that } \phi(0) = 0 \},$$

and then to prove that it is equivalent to optimize on  $\text{Quad}_0$  for problem (2.6).

**Proposition 3.1.** *Let Assumptions 1, 2 and 3 hold true, then  $|U| < \infty$ , and*

$$U = \inf_{\phi \in \text{Quad}_0} u(\phi). \quad (3.1)$$

Our second approximation is on the growth coefficient of  $\phi$  in  $\text{Quad}_0$ . Let  $K$  be a positive constant, we denote

$$U^K := \inf_{\phi \in \text{Quad}_0^K} u(\phi) \text{ with } \text{Quad}_0^K := \{ \phi \in \text{Quad}_0 : \phi(x) \leq K(|x| \vee x^2) \}. \quad (3.2)$$

By the convexity of functions in  $\text{Quad}_0$ , we see that every  $\phi \in \text{Quad}_0$  is in fact locally Lipschitz continuous, and hence  $\text{Quad}_0 = \cup_{K>0} \text{Quad}_0^K$ . Then it follows immediately that

$$U^K \searrow U \text{ as } K \rightarrow \infty. \quad (3.3)$$

The third approximation is on the tail of functions in  $\text{Quad}_0^K$ . Given a constant  $M \geq M_0$ , where  $M_0$  is given in Assumption 1, we denote

$$\text{Quad}_0^{K,M} := \{ \phi \in \text{Quad}_0^K \text{ such that } \phi(x) = Kx^2 \text{ for } |x| \geq 2M \}, \quad (3.4)$$

and

$$U^{K,M} := \inf_{\phi \in \text{Quad}_0^{K,M}} u(\phi). \quad (3.5)$$

**Proposition 3.2.** *Let Assumptions 1, 2 and 3 hold, then*

$$0 \leq U^{K,M} - U^K \leq \mu_1(\phi_{K,M}), \quad (3.6)$$

where

$$\phi_{K,M}(x) := 4KM(|x| - M)\mathbf{1}_{M \leq |x| \leq 2M} + Kx^2\mathbf{1}_{|x| > 2M}. \quad (3.7)$$

Clearly,  $\phi_{K,M} \in \text{Quad}_0^{K,M}$  and for every fixed  $K > 0$ ,  $\mu_1(\phi_{K,M}) \rightarrow 0$  as  $M \rightarrow \infty$  when  $\mu_1$  satisfies Assumption 1.

For the fourth step of the analytic approximation, we first introduce

$$\lambda^{\phi,T}(t, x) := \sup_{\tau \in \mathcal{T}^\infty, \tau \leq T-t} \mathbb{E}[g^\phi(t + \tau, x + B_\tau)], \quad \lambda_0^{\phi,T}(\cdot) := \lambda^{\phi,T}(0, \cdot),$$

$$\lambda^{\phi, \tau_R}(t, x) := \sup_{\tau \in \mathcal{T}^\infty, \tau \leq \tau_x^R} \mathbb{E}[g^\phi(t + \tau, x + B_\tau)], \quad (3.8)$$

and

$$\lambda^{\phi, T, R}(t, x) := \sup_{\tau \in \mathcal{T}^\infty, \tau \leq \tau_x^R \wedge (T-t)} \mathbb{E}[g^\phi(t + \tau, x + B_\tau)], \quad (3.9)$$

where

$$\tau_x^R := \inf\{s : x + B_s \notin (-R, R)\}.$$

**Lemma 3.3.** *Let Assumptions 2 and 3 holds true, and  $L_0, M_0$  are given in the Assumptions. Suppose that  $K > L_0, M \geq M_0$  and  $R \geq (1 + \sqrt{\frac{K}{K-L_0}})M$ . Then for every  $\phi \in \text{Quad}_0^{K, M}$ ,*

$$\lambda^\phi(t, x) = \lambda^{\phi, \tau_R}(t, x), \quad \text{and} \quad \lambda^{\phi, T}(t, x) = \lambda^{\phi, T, R}(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Given  $\phi \in \text{Quad}_0^{K, M}$ , we define

$$U^{K, M, T} := \inf_{\phi \in \text{Quad}_0^{K, M}} u^T(\phi), \quad \text{with} \quad u^T(\phi) := \mu_0(\lambda_0^{\phi, T}) + \mu_1(\phi). \quad (3.10)$$

**Proposition 3.4.** *Let Assumptions 1, 2 and 3 hold,  $M_0$  and  $L_0$  be constants given in Assumption 2,  $K > L_0, M \geq M_0, R = (1 + \sqrt{\frac{K}{K-L_0}})M$  and  $L = 2(K + 2L_0)(R^2 \vee 1)$ , we denote*

$$\delta := -\log(q(R)) > 0, \quad \text{where} \quad q(R) := \frac{1}{\sqrt{2\pi}} \int_{-2R}^{2R} e^{-x^2/2} dx.$$

Then

$$0 \leq U^{K, M} - U^{K, M, T} \leq L e^{-\delta(T-1)}. \quad (3.11)$$

Finally, we just remark that  $U^{K, M, T}$  in (3.10) is defined via  $\lambda^{\phi, T}$  which is equivalent to  $\lambda^{\phi, T, R}$  from Lemma 3.3. Then by Theorem 6.7 of Touzi [19], we can characterized  $\lambda^{\phi, T, R}$  as the viscosity solution of a variational inequality.

**Proposition 3.5.** *The function  $\lambda^{\phi, T, R}$  defined in (3.9) is the unique viscosity solution of variational inequality*

$$\min \left( \lambda - g^\phi, -\frac{1}{2} \frac{\partial^2 \lambda}{\partial x^2} - \frac{\partial \lambda}{\partial t} \right)(t, x) = 0, \quad \text{on} \quad [0, T) \times (-R, R), \quad (3.12)$$

with boundary condition

$$\lambda(t, x) = g^\phi(t, x), \quad \text{on} \quad ([0, T] \times \{\pm R\}) \cup (\{T\} \times [-R, R]).$$

### 3.2 A first analysis

Before proving the convergence results given in Propositions 3.1, 3.2 and 3.4, we first give two well-known properties of the stopping times on a Brownian motion and report their proofs for completeness. We then provide also a first analysis on  $u(\phi)$  and  $U$  in (2.6).

**Lemma 3.6.** *Let  $\psi : (t, x) \in \mathbb{R}^+ \times \mathbb{R} \mapsto \psi(t, x) \in \mathbb{R}$  be a function Lipschitz in  $t$  and satisfying  $\sup_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}} \frac{|\psi(t,x)|}{1+x^2} < \infty$ . Then for every  $\tau \in \mathcal{T}^\infty$ ,*

$$\mathbb{E} [\psi(\tau, B_\tau)] = \lim_{t \rightarrow \infty} \mathbb{E} [\psi(\tau \wedge t, B_{\tau \wedge t})]. \quad (3.13)$$

In particular,

$$\mathbb{E}[B_\tau^2] = \lim_{t \rightarrow \infty} \mathbb{E}[B_{\tau \wedge t}^2] = \lim_{t \rightarrow \infty} \mathbb{E}[\tau \wedge t] = \mathbb{E}[\tau] \quad \text{and} \quad \mathbb{E}[B_\tau] = 0. \quad (3.14)$$

**Proof.** Given a stopping time  $\tau \in \mathcal{T}^\infty$ , let  $Y_t := B_{\tau \wedge t}$ . Then by assumptions on  $\psi$ , there is a constant  $C > 0$  such that

$$\psi(B_{\tau \wedge t}, \tau \wedge t) \leq C(1 + Y_t^2 + \tau) \leq C\left(1 + \sup_{s \geq 0} Y_s^2 + \tau\right), \quad \forall t \geq 0.$$

We notice that  $(Y_t)_{t \geq 0}$  is a continuous uniformly integrable martingale by its definition, and  $\mathbb{E}[\sup_{s \geq 0} Y_s^2] \leq 4\mathbb{E}[\tau] < \infty$  by Doob's inequality. And hence it follows by the dominated convergence theorem that (3.13) holds true.  $\square$

Given  $T > 0$ , we denote by  $\mathcal{T}^T$  the collection of all  $\mathbb{F}$ -stopping times taking value in  $[0, T]$ , i.e.

$$\mathcal{T}^T := \{\tau \wedge T : \tau \in \mathcal{T}^\infty\}. \quad (3.15)$$

**Lemma 3.7.** *Let  $\psi \in \text{Quad}$  and denote by  $\psi^{\text{conv}}$  its convex envelope, then*

$$\inf_{\tau \in \mathcal{T}^T} \mathbb{E} \psi(B_\tau) \rightarrow \inf_{\tau \in \mathcal{T}^\infty} \mathbb{E} \psi(B_\tau) = \psi^{\text{conv}}(0), \quad \text{as } T \rightarrow \infty.$$

**Proof.** Let  $a \leq 0 \leq b$  be two constants and  $\tau_{a,b} := \inf\{t : B_t \notin (a, b)\}$ . We first notice that  $\tau_{a,b} \in \mathcal{T}^\infty$  since  $\mathbb{E}[\tau_{a,b}] = \lim_{t \rightarrow \infty} \mathbb{E}[\tau_{a,b} \wedge t] = \lim_{t \rightarrow \infty} \mathbb{E}[B_{\tau_{a,b} \wedge t}^2] \leq (a^2 + b^2) < \infty$ . Hence by (3.14),  $\mathbb{E}[B_{\tau_{a,b}}] = 0$ , which implies that  $\mathbb{P}(B_{\tau_{a,b}} = a) = \frac{b}{b-a}$  and  $\mathbb{P}(B_{\tau_{a,b}} = b) = \frac{-a}{b-a}$ . Therefore,

$$\inf_{\tau \in \mathcal{T}^\infty} \mathbb{E} \psi(B_\tau) \leq \inf_{a < 0 < b} \mathbb{E} \psi(B_{\tau_{a,b}}) = \inf_{a < 0 < b} \left( \frac{b}{b-a} \psi(a) + \frac{-a}{b-a} \psi(b) \right) = \psi^{\text{conv}}(0).$$

On the other side, for every  $\tau \in \mathcal{T}^\infty$ , by Jensen's inequality together with the fact that  $\mathbb{E}[B_\tau] = 0$  from (3.14), it follows that  $\psi^{\text{conv}}(x) \leq \mathbb{E}[\psi^{\text{conv}}(x + B_\tau)] \leq \mathbb{E}[\psi(x + B_\tau)]$ , and therefore,

$$\inf_{\tau \in \mathcal{T}^\infty} \mathbb{E} \psi(B_\tau) = \psi^{\text{conv}}(0).$$

Finally, the convergence of  $\inf_{\tau \in \mathcal{T}^T} \mathbb{E} \psi(B_\tau)$  to  $\inf_{\tau \in \mathcal{T}^\infty} \mathbb{E} \psi(B_\tau)$  as  $T \rightarrow \infty$  is a direct consequence of (3.13) in Lemma 3.6.  $\square$

With the above two lemmas, we can now give a first analysis on  $u(\phi)$  as well as  $U$  defined in (2.6).

**Corollary 3.8.** *Let  $\phi \in \text{Quad}$  and  $(a, b) \in \mathbb{R}^2$ , then  $u(\phi) = u(\phi_{a,b})$ , where  $\phi_{a,b}$  is given by  $\phi_{a,b}(x) := \phi(x) + ax + b$ .*

**Proof.** By the definition of  $\lambda_0^\phi$  in (2.5) together with Lemma 3.6, it follows that  $\lambda_0^{\phi_{a,b}}(x) = \lambda_0^\phi(x) + ax + b$ . Moreover, as discussed in Remark 2.1,  $\mu_0(I) = \mu_1(I)$  for the identity function  $I$ . Then we get  $u(\phi) = u(\phi_{a,b})$  by their definitions in (2.6).  $\square$

The next result can be viewed as a consequence of Dupire's [7] observation that *variance swap* is equivalent to a European option with payoff function  $g(x) = x^2$ . We give it in our context.

**Corollary 3.9.** *Let Assumptions 1, 2 hold true,  $\psi \in \text{Quad}$ ,  $K \in \mathbb{R}$  and  $g(t, x)$  be the payoff function, we define another payoff function  $g_{K, \psi}$  by  $g_{K, \psi}(t, x) := g(t, x) + Kt + \psi(x)$ . Denote by  $U(g)$  (resp.  $U(g_{K, \psi})$ ) the no-arbitrage price upper bound defined in (2.6) associated with the payoff function  $g$  (resp.  $g_{K, \psi}$ ). Then*

$$U(g_{K, \psi}) = U(g) + KC_0 + \mu_1(\psi), \quad (3.16)$$

where  $C_0$  is given by (2.2). In particular, the upper bound of “variance swap” option is  $C_0$ , and the bound of a European option with payoff function  $\psi(x)$  is given by  $\mu_1(\psi)$ .

**Proof.** Given  $\phi \in \text{Quad}$ , we denote  $\phi_{K, \psi}(x) := \phi(x) + \psi(x) + Kx^2$  which also belongs to  $\text{Quad}$ , then by (3.14)

$$\mathbb{E}[g_{K, \psi}(t + \tau, x + B_\tau) - \phi_{K, \psi}(x + B_\tau)] = \mathbb{E}[g^\phi(t + \tau, x + B_\tau)] - Kx^2, \quad \forall \tau \in \mathcal{T}^\infty.$$

It follows by the definition of  $U$  in (2.6) that  $U(g_{K, \psi}) \geq U(g) + KC_0 + \mu_1(\psi)$ . And moreover, by the arbitrariness of  $K \in \mathbb{R}$ ,  $\psi \in \text{Quad}$  and symmetric relationship between  $g$  and  $g_{K, \psi}$ , we proved (3.16).

For the last statement, it follows by (3.16) that we only need to prove that  $U(g^0) = 0$  with  $g^0 \equiv 0$ . Indeed, with the payoff function  $g^0 \equiv 0$ , we get immediately from (2.5) and (2.6) as well as Lemma 3.7 that

$$u(\phi) = -\mu_0(\phi^{\text{conv}}) + \mu_1(\phi) \geq \mu_1(\phi^{\text{conv}}) - \mu_0(\phi^{\text{conv}}) \geq 0,$$

where the last inequality comes from Assumption 1. Finally, we conclude with  $U(g^0) = 0$  by the fact that  $u(g^0) = 0$ .  $\square$

**Remark 3.10.** *Let us consider the formulation of  $\bar{U}$  in (1.5). From the definition of  $\mathcal{T}$  in (1.6), we see that every stopping time  $\tau \in \mathcal{T}$  conditioned on  $W_0$  belongs to  $\mathcal{T}^\infty$  defined in (2.3). Then by the same arguments, we have under the same conditions as in Corollary 3.9 that*

$$\bar{U}(g_{K, \psi}) = \bar{U}(g) + KC_0 + \mu_1(\psi),$$

where  $\bar{U}(g)$  (resp.  $\bar{U}(g_{K, \psi})$ ) denotes the price bound associated with payoff function  $g$  (resp.  $g_{K, \psi}$ ) given in (1.5).

### 3.3 Proofs of the convergence

Now we are ready to give the proof of the convergence results in Propositions 3.1, 3.2 and 3.4.

**Proof of Proposition 3.1.** First, with the positive constant  $L_0$  given in Assumption 1, we have

$$g(0, x) \leq g(t, x) \leq g(0, x) + L_0 t.$$

Moreover, it is clear that  $U$  is monotone w.r.t. the payoff function  $g$  by its definition in (2.6). Then it follows by Corollary 3.9 that

$$\mu_1(g(0, \cdot)) \leq U \leq \mu_1(g(0, \cdot)) + L_0 C_0, \quad \text{with } C_0 \text{ defined in (2.2).}$$

Next, let us prove the equality (3.1) for  $U$ . Let  $T \in \mathbb{R}^+$ ,  $\tau_0 \in \mathcal{T}^T$  and  $\phi \in \text{Quad}$ . By the dominated convergence theorem, it is easy to see that  $x \mapsto \inf_{\tau \in \mathcal{T}^T} \mathbb{E}\phi(x + B_\tau)$  is continuous.

This, together with the weak dynamic programming in Theorem 4.1 of Bouchard and Touzi [4], implies the dynamic programming principle:

$$\inf_{\tau_0 \leq \tau \leq T} \mathbb{E} \phi(x + B_\tau) = \mathbb{E} \left[ \inf_{\tau_0 \leq \tau \leq T} \mathbb{E} [\phi(x + B_\tau) | \mathcal{F}_{\tau_0}] \right].$$

Then for constants  $\hat{T} > T$ ,

$$\lambda_0^\phi(x) = \sup_{\tau \in \mathcal{T}^\infty} \mathbb{E} [g^\phi(\tau, x + B_\tau)] \geq \sup_{\tau_0 \leq \tau \leq \hat{T}} \mathbb{E} [g(\tau, x + B_\tau) - \phi(x + B_\tau)].$$

By the increase of  $g$  in  $t$  and its convexity in  $x$  from Assumption 3, we have

$$\mathbb{E} [g(\tau, x + B_\tau) | \mathcal{F}_{\tau_0}] \geq \mathbb{E} [g(\tau_0, x + B_\tau) | \mathcal{F}_{\tau_0}] \geq g(\tau_0, x + B_{\tau_0}),$$

and hence

$$\lambda_0^\phi(x) \geq \mathbb{E} [g(\tau_0, x + B_{\tau_0})] - \mathbb{E} \left[ \inf_{\tau_0 \leq \tau \leq \hat{T}} \mathbb{E} [\phi(x + B_\tau) | \mathcal{F}_{\tau_0}] \right].$$

Sending  $\hat{T}$  to  $+\infty$ , by Lemma 3.7, it follows that

$$\lambda_0^\phi(x) \geq \mathbb{E} [g(\tau_0, x + B_{\tau_0}) - \phi^{conv}(x + B_{\tau_0})].$$

Thus, by arbitrariness of  $\tau_0$  in  $\mathcal{T}^T$  as well as that of  $T \in \mathbb{R}^+$ , we get

$$\begin{aligned} \lambda_0^\phi(x) &\geq \lim_{T \rightarrow \infty} \sup_{\tau_0 \in \mathcal{T}^T} \mathbb{E} [g(\tau_0, x + B_{\tau_0}) - \phi^{conv}(x + B_{\tau_0})], \\ &= \sup_{\tau_0 \in \mathcal{T}^\infty} \mathbb{E} [g(\tau_0, x + B_{\tau_0}) - \phi^{conv}(x + B_{\tau_0})], \end{aligned}$$

where the last equality is a direct consequence of Lemma 3.6 since  $\phi^{conv}$  is either of quadratic growth or equals to  $-\infty$ .

Finally, since  $\phi \geq \phi^{conv}$ , by the definition of  $u$  and  $U$  in (2.6), it is clear that the infimum in (2.6) can be taken on the collection of all convex functions in Quad. Moreover, by the property of  $u(\phi)$  in Corollary 3.8, the infimum can be then taken on the collection of all positive convex functions  $\phi$  in Quad such that  $\phi(0) = 0$ , i.e.  $U = \inf_{\phi \in \text{Quad}_0} u(\phi)$ . We then proved (3.1).  $\square$

**Proof of Proposition 3.2.** Let us first recall that every function  $\phi \in \text{Quad}_0^K$  is nonnegative, convex such that  $\phi(0) = 0$  and  $\phi(x) \leq K(|x| \vee x^2)$ . Given  $\phi \in \text{Quad}_0^K$ , we denote  $\phi_M := \phi \vee \phi_{K,M}$ . Clearly,  $\phi_M$  lies in  $\text{Quad}_0^{K,M}$  and  $\lambda^{\phi_M} \leq \lambda^\phi$  since  $\phi_M \geq \phi$ . It follows from the definition of  $u(\phi)$  in (2.6) and positivity of  $\phi$  that

$$u(\phi_M) - u(\phi) \leq \mu_1(\phi_M) - \mu_1(\phi) \leq \mu_1(\phi_{K,M}).$$

This, together with the arbitrariness of  $\phi \in \text{Quad}_0^K$  and the fact that  $\phi_M \in \text{Quad}_0^{K,M}$ , concludes the proof for (3.6).  $\square$

In preparation of the proof for Lemma 3.3 and Proposition 3.4, we first give a property for functions in  $\text{Quad}_0^{K,M}$ .

**Lemma 3.11.** *Let Assumptions 2 and 3 hold true,  $L_0, M_0$  be the constants given in Assumption 2,  $K > L_0, M \geq M_0$  and  $R = (1 + \sqrt{\frac{K}{K-L_0}})M$ . Given fixed  $t \in \mathbb{R}^+$  and  $\phi \in \text{Quad}_0^{K,M}$ , we denote*

$$\psi(x) := -g^\phi(t, x) - L_0 x^2 = \phi(x) - g(t, x) - L_0 x^2.$$

*Then  $\psi^{conv}(x) = \psi(x)$  when  $x \notin [-R, R]$ .*

**Proof.** By Assumption 2, we know that there are constants  $C_1, C_2$  such that  $x \mapsto g(t, x)$  is affine with derivative  $C_1$  when  $x \geq M$ , and affine with derivative  $C_2$  when  $x \leq -M$ . For fixed  $t \in \mathbb{R}^+$ , let  $\chi$  be a continuous function defined on  $\mathbb{R}$  by the following:  $\chi$  is affine on intervals  $[-2M, -M]$ ,  $[-M, 0]$ ,  $[0, M]$ ,  $[M, 2M]$  and

$$\begin{cases} \chi(0) & := -g(t, 0), \\ \chi(\pm M) & := -L_0 M^2 - g(t, \pm M), \\ \chi(\pm 2M) & := 4(K - L_0)M^2 - g(t, \pm 2M), \\ \chi(x) & := (K - L_0)x^2 - g(t, 2M) - C_1(x - 2M), & x \geq 2M, \\ \chi(x) & := (K - L_0)x^2 - g(t, -2M) - C_2(x + 2M), & x \leq -2M. \end{cases}$$

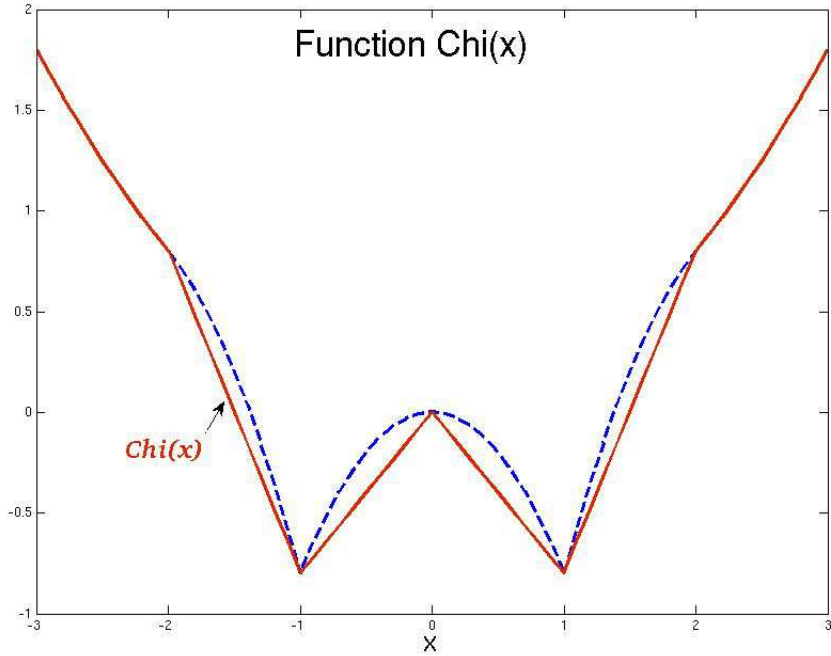


Figure 1: An example of function  $\chi$  when  $M = 1$ .

By Assumptions 2 and 3, we can verify that for every  $\phi \in \text{Quad}_0^{K,M}$  and the corresponding  $\psi$  defined in the statement of the lemma,

$$\psi(x) \begin{cases} \geq \chi(x), & \text{when } x \in [-2M, 2M], \\ = \chi(x), & \text{when } x \notin [-2M, 2M]. \end{cases}$$

Then given  $x \notin [-R, R]$ , it follows by a simple calculation that  $\chi(y) \geq \chi(x) + \chi'(x)(y - x)$  for every  $y \in \mathbb{R}$ , which implies that  $\chi^{\text{conv}}(x) = \chi(x)$ . And hence  $\psi(x) \geq \psi^{\text{conv}}(x) \geq \chi^{\text{conv}}(x) = \chi(x) = \psi(x)$  for  $x \notin [-R, R]$ .  $\square$

**Proof of Lemma 3.3.** We shall just show that  $\lambda^\phi = \lambda^{\phi, \tau_R}$ , since  $\lambda^{\phi, T} = \lambda^{\phi, T, R}$  holds with the same arguments. Moreover, to prove  $\lambda^\phi = \lambda^{\phi, \tau_R}$ , it is enough to show that  $\lambda^\phi \leq \lambda^{\phi, \tau_R}$  since its inverse inequality is obvious from the definition of  $\lambda^{\phi, \tau_R}$  in (3.8).

First, let us fix  $t \in \mathbb{R}^+$  and  $x \notin (-R, R)$ , we denote  $\psi_x(y) := -g^\phi(t, y) - L_0 y^2 + L_0 x^2$ . Then by Lemma 3.11, we have  $\psi_x^{\text{conv}}(x) = \psi_x(x) = -g^\phi(t, x)$ . And it follows that for every  $\tau \in \mathcal{T}^\infty$ ,

$$\begin{aligned} \mathbb{E} [ g^\phi(t + \tau, x + B_\tau) ] &\leq \mathbb{E} [ g^\phi(t, x + B_\tau) + L_0 \tau ] \\ &= \mathbb{E} [ g^\phi(t, x + B_\tau) + L_0(x + B_\tau)^2 - L_0 x^2 ] \\ &= -\mathbb{E} \psi_x(x + B_\tau) \leq -\psi_x^{\text{conv}}(x) = g^\phi(t, x), \end{aligned} \quad (3.17)$$

which implies that  $\lambda^\phi(t, x) \leq \lambda^{\phi, \tau_R}(t, x)$  for every  $x \notin (-R, R)$  since in this case  $\tau_x^R = 0$ . Next, for every  $\tau \in \mathcal{T}^\infty$  and  $x \in [-R, R]$ , we have according to (3.17) that

$$\begin{aligned} &\mathbb{E} [ g^\phi(t + \tau, x + B_\tau) ] \\ &= \mathbb{E} [ g^\phi(t + \tau, x + B_\tau) \mathbf{1}_{\tau \leq \tau_x^R} ] + \mathbb{E} [ \mathbb{E} [ g^\phi(t + \tau, x + B_\tau) \mathbf{1}_{\tau > \tau_x^R} \mid \mathcal{F}_{\tau \wedge \tau_x^R} ] ] \\ &\leq \mathbb{E} [ g^\phi(t + \tau \wedge \tau_x^R, x + B_{\tau \wedge \tau_x^R}) ], \end{aligned}$$

which implies that  $\lambda^\phi(t, x) \leq \lambda^{\phi, \tau_R}(t, x)$  for all  $x \in [-R, R]$ .  $\square$

**Proof of Proposition 3.4.** We first derive an estimate on stopping times inferior to  $\tau_x^R$ , borrowed from Carlier and Galichon's [5] Lemma 5.2. Let  $x \in [-R, R]$ , then for every stopping time  $\tau \leq \tau_x^R$ , we have

$$\mathbb{P}(\tau \geq T) \leq \mathbb{P}(\tau_x^R \geq T) \leq \mathbb{P}_{1 \leq n \leq T}(|B_n - B_{n-1}| \leq 2R) \leq e^{-\delta(T-1)}. \quad (3.18)$$

Recall that  $\mathbb{E}[(x + B_\tau)^2] = x^2 + \mathbb{E}[\tau]$ ,  $\forall \tau \leq \tau_x^R$  from (3.14). Then by the definitions of  $\lambda^{\phi, \tau_R}$  and  $\lambda^{\phi, T, R}$  in (3.9), for every  $\phi \in \text{Quad}_0^{K, M}$ ,

$$\begin{aligned} \lambda^{\phi, \tau_R}(0, x) - \lambda^{\phi, T, R}(0, x) &\leq \sup_{\tau \leq \tau_x^R} \mathbb{E} [ g^\phi(\tau, x + B_\tau) - g^\phi(\tau \wedge T, x + B_{\tau \wedge T}) ] \\ &= \sup_{\tau \leq \tau_x^R} \mathbb{E} [ \psi(\tau \wedge T, x + B_{\tau \wedge T}) - \psi(x + B_\tau, \tau) ], \end{aligned}$$

where  $\psi(t, x) := -g^\phi(t, x) - L_0 x^2 + L_0 t$ . Clearly,  $\psi$  increases in  $t$  and  $|\psi(t, x_1) - \psi(t, x_2)| \leq 2(K + 2L_0)(R^2 \vee 1)$ ,  $\forall x_1, x_2 \in [-R, R]$  by Assumptions 2 and 3, therefore,

$$\begin{aligned} \lambda^{\phi, \tau_R}(0, x) - \lambda^{\phi, T, R}(0, x) &\leq \sup_{\tau \leq \tau_x^R} \mathbb{E} [ |\psi(\tau \wedge T, x + B_{\tau \wedge T}) - \psi(\tau \wedge T, x + B_\tau)| ] \\ &= \sup_{\tau \leq \tau_x^R} \mathbb{E} [ |\psi(T, x + B_T) - \psi(T, x + B_\tau)| \mathbf{1}_{\tau \geq T} ] \\ &\leq \sup_{\tau \leq \tau_x^R} 2(K + 2L_0)(R^2 \vee 1) \mathbb{P}(\tau \geq T) \\ &\leq L e^{-\delta(T-1)}, \end{aligned}$$

where the last inequality is from (3.18). Finally, by arbitrariness of  $\phi \in \text{Quad}_0^{K, M}$  together with Lemma 3.3, we prove (3.11).  $\square$

## 4 The numerical approximation

We shall propose a numerical method to approximate  $U^{K, M, T}$ . The idea is to compute  $\lambda^{\phi, T, R}$  with a finite differences numerical scheme, and then solve the minimization problem (3.10) with an iterative algorithm. Concretely, we shall first propose a discrete system characterized by  $h = (\Delta t, \Delta x)$ , on which there is a discrete optimization problem with value  $U_h^{K, M, T}$  close to  $U^{K, M, T}$ . Then we use the gradient projection algorithm to solve the discrete optimization problem of  $U_h^{K, M, T}$ .



### 4.1 A finite difference approximation

Let  $T, R > 2M$  be constants in  $\mathbb{R}^+$  and  $(l, r, m) \in \mathbb{N}^3$ ,  $h = (\Delta x, \Delta t) \in (\mathbb{R}^+)^2$  such that  $l\Delta t = T$ ,  $r\Delta x = R$  and  $m\Delta x = M$ . Denote  $x_i = i\Delta x$  and  $t_k = k\Delta t$  and define the discrete grid:

$$\mathcal{N} := \{x_i : i \in \mathbb{Z}\}, \quad \mathcal{N}_R := \mathcal{N} \cap [-R, R],$$

$$\mathcal{M}_{T,R} := \{(t_k, x_i) : (k, i) \in \mathbb{Z}^+ \times \mathbb{Z}\} \cap ([0, T] \times [-R, R]),$$

The terminal set, boundary set as well as interior set of  $\mathcal{M}_{T,R}$  are denoted by

$$\partial_T \mathcal{M}_{T,R} := \{(T, x_i) : -r \leq i \leq r\}, \quad \partial_R \mathcal{M}_{T,R} := \{(t_k, \pm R) : 0 \leq k \leq l\},$$

$$\mathring{\mathcal{M}}_{T,R} := \mathcal{M}_{T,R} \setminus (\partial_R \mathcal{M}_{T,R} \cup \partial_T \mathcal{M}_{T,R}).$$

Given a function  $w(t, x)$  defined on  $\mathcal{M}_{T,R}$ , we introduce the discrete derivative of  $w$ :

$$D^2 w(t_k, x_i) := \frac{w(t_k, x_{i+1}) - 2w(t_k, x_i) + w(t_k, x_{i-1}))}{\Delta x^2}.$$

Then with function  $\varphi$  defined on  $\mathcal{N}_R$  and the notation

$$g^\varphi(t_k, x_i) := g(t_k, x_i) - \varphi(x_i) \tag{4.1}$$

as well as  $\theta \in [0, 1]$ , we define  $\lambda_h^{\varphi, T, R}$  as the solution of the finite difference scheme of variational inequality (3.12) on  $\mathcal{M}_{T,R}$ :

$$\begin{cases} \lambda_h^{T,R}(t_{k+1}, x_i) - \tilde{\lambda}_h^{T,R}(t_k, x_i) \\ \quad + \frac{1}{2}\Delta t \left( \theta D^2 \tilde{\lambda}_h^{T,R}(t_k, x_i) + (1-\theta) D^2 \lambda_h^{T,R}(t_{k+1}, x_i) \right) = 0, \\ \lambda_h^{T,R}(t_k, x_i) = \max \left( g^\varphi(t_k, x_i), \tilde{\lambda}_h^{T,R}(t_k, x_i) \right), & (t_k, x_i) \in \mathring{\mathcal{M}}_{T,R}, \\ \lambda_h^{T,R}(t_k, x_i) = g^\varphi(t_k, x_i), & (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}. \end{cases} \tag{4.2}$$

We notice that the above  $\theta$ -scheme has clearly a unique solution. And it is a consistent scheme for (3.12) in sense of Barles and Souganidis [2]. To see this, it is enough to rewrite the second equation of (4.2) as

$$\min \left( \lambda_h^{T,R} - g^\varphi, \frac{\lambda_h^{T,R} - \tilde{\lambda}_h^{T,R}}{\Delta t} \right) (t_k, x_i) = 0$$

We shall assume in addition that the discretization parameters  $h = (\Delta t, \Delta x)$  satisfy the CFL condition

$$(1-\theta) \frac{\Delta t}{\Delta x^2} \leq 1. \tag{4.3}$$

Then the finite difference scheme (4.2) is monotone in sense of [2], and the numerical solution  $\lambda_h^{\varphi, T, R}$  converges to  $\lambda^{\phi, T, R}$  given  $\varphi := \phi|_{\mathcal{N}}$  by the results of [2].

**Remark 4.1.** *The discrete system (4.2) is the  $\theta$ -scheme for variational inequality (3.12) with Dirichlet boundary condition  $g(x, t) - \varphi(x)$  on  $\partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}$ . It is well-known that when the finite difference scheme is explicit (i.e.  $\theta = 0$ ) and the CFL condition  $\frac{\Delta t}{\Delta x^2} \leq 1$  holds, it can be interpreted as the dynamic programming principle for a system on a Markov chain  $\Lambda$  (see e.g. Kushner [14]). This interpretation holds also true for general  $\theta$ -scheme, as we shall see later in the proof of Proposition 4.4.*

We next introduce a natural approximation of  $u_T(\phi)$  in (3.10):

$$u_{h,T}(\varphi) := \mu_0(\text{lin}^R[\lambda_{h,0}^{\varphi,T,R}]) + \mu_1(\text{lin}^R[\varphi]), \quad (4.4)$$

where  $\lambda_{h,0}^{\varphi,T,R}(\cdot) := \lambda_h^{\varphi,T,R}(0, \cdot)$ , and for every function  $\varphi$  defined on  $\mathcal{N}_R$ , we denote by  $\text{lin}^R[\varphi]$  its linear interpolation extended by zero outside  $[-R, R]$ .

**Assumption 4.** *There are constants  $(\rho_1, \rho_2, L_{K,M,T}) \in (\mathbb{R}^+)^3$  which are independent of  $h = (\Delta t, \Delta x)$  such that*

$$\mu_0 \left( \left| \lambda_0^{\phi,T,R} \mathbf{1}_{[-R,R]} - \text{lin}^R[\lambda_{h,0}^{\phi,T,R}] \right| \right) \leq L_{K,M,T} (\Delta x^{\rho_1} + \Delta t^{\rho_2}), \quad (4.5)$$

*for every  $\phi \in \text{Quad}_0^{K,M}$  and  $\varphi = \phi|_{\mathcal{N}_R}$ .*

**Remark 4.2.** *When  $\theta = 1$ , (4.2) is the implicit scheme for (3.12), then Assumption 4 holds true with  $\rho_1 = \frac{1}{2}$  and  $\rho_2 = \frac{1}{4}$  in spirit of the analysis of Krylov [13].*

*When  $\theta = 0$  and the CFL condition (4.3) is true, (4.2) is a monotone explicit scheme, then in spirit of Barles and Jakobsen [1], Assumption 4 holds with  $\rho_1 = \frac{1}{10}$  and  $\rho_2 = \frac{1}{5}$ .*

Let  $\text{Quad}_{0,h}^{K,M}$  be the collection of all functions on the grid  $\mathcal{N}_R$  defined as restrictions of functions in  $\text{Quad}_0^{K,M}$ :

$$\text{Quad}_{0,h}^{K,M} := \{ \varphi := \phi|_{\mathcal{N}_R} \text{ for some } \phi \in \text{Quad}_0^{K,M} \}. \quad (4.6)$$

We can then provide a discrete approximation for  $U^{K,M,T}$  in (3.10):

$$U_h^{K,M,T} := \inf_{\varphi \in \text{Quad}_{0,h}^{K,M}} u_{h,T}(\varphi). \quad (4.7)$$

Let  $B(\mathcal{N}_R)$  be the set of all bounded functions defined on the grid  $\mathcal{N}_R$ , then clearly

$$\text{Quad}_{0,h}^{K,M} = \left\{ \varphi \in B(\mathcal{N}_R) \text{ nonnegative, convex satisfying } \varphi(0) = 0, \varphi(x_i) = Kx_i^2, \right. \\ \left. \text{for all } 2m \leq |i| \leq r, \text{ and } |\varphi(x_{i+1}) - \varphi(x_i)| \leq 4KM\Delta x, \forall -2m < i \leq 2m \right\}. \quad (4.8)$$

**Proposition 4.3.** *Let Assumptions 2, 4 hold, then with the same constants  $L_{K,M,T}, \rho_1, \rho_2$  introduced in Assumption 4,*

$$\left| U^{K,M,T} - U_h^{K,M,T} \right| \leq L_{K,M,T} (\Delta x^{\rho_1} + \Delta t^{\rho_2}) + 4KR\Delta x + (\mu_0 + \mu_1)(\phi_K^R), \quad (4.9)$$

where  $\phi_K^R(x) := Kx^2 \mathbf{1}_{|x| > R}$ .

**Proof.** First, given  $\phi \in \text{Quad}_0^{K,M}$  which is  $4KR$ -Lipschitz, we introduce  $\varphi := \phi|_{\mathcal{N}_R} \in \text{Quad}_{0,h}^{K,M}$  so that  $|\text{lin}^R[\varphi] - \phi|_{L^\infty([-R,R])} \leq 4KR\Delta x$ . Then it follows by Assumption 4 that  $|u_T(\phi) - u_{h,T}(\varphi)| \leq L_{K,M,T}(\Delta x^{\rho_1} + \Delta t^{\rho_2}) + 4KR\Delta x + (\mu_0 + \mu_1)(\phi_K^R)$ , and hence

$$U^{K,M,T} - U_h^{K,M,T} \leq L_{K,M,T} (\Delta x^{\rho_1} + \Delta t^{\rho_2}) + 4KR\Delta x + (\mu_0 + \mu_1)(\phi_K^R).$$

Next, given  $\varphi \in \text{Quad}_{0,h}^{K,M}$ , we take  $\phi := \text{lin}^R[\varphi] + \phi_K^R \in \text{Quad}_0^{K,M}$ . It follows by Assumption 4 that  $|u_T(\phi) - u_{h,T}(\varphi)| \leq L_{K,M,T}(\Delta x^{\rho_1} + \Delta t^{\rho_2}) + (\mu_0 + \mu_1)(\phi_K^R)$ , and therefore,

$$U_h^{K,M,T} - U^{K,M,T} \leq L_{K,M,T} (\Delta x^{\rho_1} + \Delta t^{\rho_2}) + (\mu_0 + \mu_1)(\phi_K^R).$$

□

## 4.2 Gradient projection algorithm

As we can easily observe from its definition in (2.6) that  $\phi \mapsto u(\phi)$  is convex since it is represented as the supremum of a family of linear map, we shall show that  $\varphi \mapsto u_{h,T}(\varphi)$  is also convex, then a natural candidate for the resolution of  $U_h^{K,M,T} = \inf_{\varphi \in \text{Quad}_{0,h}^{K,M}} u_{h,T}(\varphi)$  in (4.7) is the gradient projection algorithm. Recall that  $B(\mathcal{N}_R)$  denotes the collection of all bounded function on  $\mathcal{N}_R$ .

**Proposition 4.4.** *Under the CFL condition (4.3), the function  $\varphi \mapsto u_{h,T}(\varphi)$  is convex.*

**Proof.** Let us first rewrite the finite differences scheme (4.2) into a vector system. Denote  $\alpha := \frac{\Delta t}{2\Delta x^2}$ ,  $\lambda_k := (\lambda_h^{\varphi,T,R}(t_k, x_i))_{-r \leq i \leq r}$ ,  $\tilde{\lambda}_k := (\tilde{\lambda}_h^{\varphi,T,R}(t_k, x_i))_{-r \leq i \leq r}$  and  $q_k := (g^\varphi(t_k, x_i))_{-r \leq i \leq r} \in \mathbb{R}^{2r+1}$ . Let  $I_{2r+1}$  denote the  $(2r+1) \times (2r+1)$  identity matrix,  $\Pi$  and  $b_k \in \mathbb{R}^{2r+1}$  be defined by

$$\Pi := \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 \\ 1 & -2 & 1 & 0 & & \\ 0 & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 & 0 \\ 0 & & & 0 & 1 & -2 & 1 \\ 0 & & & 0 & 0 & 0 & 0 \end{pmatrix}, \quad b_k := \begin{pmatrix} q_k(-r) - \lambda_{k+1}(-r) \\ 0 \\ \vdots \\ 0 \\ q_k(r) - \lambda_{k+1}(r) \end{pmatrix},$$

and  $\Theta := [I_{2r+1} - \theta\alpha\Pi]^{-1}[I_{2r+1} + (1-\theta)\alpha\Pi]$ , then scheme (4.2) can be rewritten as

$$\tilde{\lambda}_k = \Theta\lambda_{k+1} + b_k, \quad \text{and} \quad \lambda_k = \tilde{\lambda}_k \vee q_k. \quad (4.10)$$

Under CFL condition (4.3), we can verify that the above scheme is monotone, i.e. every element of  $\Theta$  is positive, and moreover,  $\Theta\mathbf{1} = \mathbf{1}$ , where  $\mathbf{1} := (1, \dots, 1)^T \in \mathbb{R}^{2r+1}$ . It follows that  $\Theta$  can be the probability transition matrix of some Markov chain  $\Lambda$ , whose state space is the grid  $\mathcal{N}_R$  with absorbing boundary. Let  $\mathcal{T}_h^R$  denote the collection of all stopping times  $\tau$  on  $\Lambda$  such that  $\Lambda_t \in \mathcal{N}_R$  for  $t \leq \tau$ , then  $\lambda_h^{\varphi,T,R}$  can be represented as solutions of an optimal stopping problem on  $\Lambda$ :

$$\lambda_h^{\varphi,T,R}(t_k, x_i) = \sup_{\tau \in \mathcal{T}_h^R, \tau \geq t_k} \mathbb{E} [g^\varphi(\Lambda_\tau, \tau) \mid \Lambda_{t_k} = x_i].$$

Now given a family of stopping times  $\tau_h = (\tau_h^i)_{-r \leq i \leq r}$  in  $\mathcal{T}_h^R$ , we introduce the function  $\lambda_{h,0}^{\varphi,T,R,\tau_h}$  defined on  $\mathcal{N}_R$ :

$$\lambda_{h,0}^{\varphi,T,R,\tau_h}(x_i) := \mathbb{E} [g^\varphi(\Lambda_\tau, \tau) \mid \Lambda_0 = x_i].$$

Then  $u_{h,T}$  has an equivalent representation:

$$u_{h,T}(\varphi) = \sup_{\tau_h \in (\mathcal{T}_h^R)^{2r+1}} \bar{u}_{h,T}^{\tau_h}(\varphi) := \sup_{\tau_h \in (\mathcal{T}_h^R)^{2r+1}} \mu_0(\text{lin}^R[\lambda_{h,0}^{\varphi,T,R,\tau_h}]) + \mu_1(\text{lin}^R[\varphi]). \quad (4.11)$$

Clearly, for every  $\tau_h$ ,  $\varphi \mapsto \bar{u}_{h,T}^{\tau_h}(\varphi)$  is linear, and finally it follows by (4.11) that  $\varphi \mapsto u_{h,T}(\varphi)$  is convex.  $\square$

**Remark 4.5.** *In the above Markov chain system (4.11), given  $\varphi \in B(\mathcal{N}_R)$ , let us define an optimal stopping time  $\tau_h(\varphi)$  by*

$$\tau_h(\varphi) := \inf \{ t_k : \lambda_h^{\varphi,T,R,\tau_h}(t_k, \Lambda_{t_k}) = g^\varphi(t_k, \Lambda_{t_k}) \}, \quad (4.12)$$

then clearly,

$$u_{h,T}(\varphi) = \sup_{\tau_h \in (\mathcal{T}_R^h)^{2r+1}} \bar{u}_{h,T}^{\tau_h}(\varphi) = \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi). \quad (4.13)$$

Now we are ready to give the gradient projection algorithm for  $U_h^{K,M,T}$  in (4.7). Given  $\varphi \in B(\mathcal{N}_R)$ , we denote by  $P_{\text{Quad}_{0,h}^{K,M}}[\varphi]$  its projection on  $\text{Quad}_{0,h}^{K,M}$ . Of course, such a projection depends on the norm equipped on  $B(\mathcal{N}_R)$ , which is an important issue to be discussed later. Let  $\gamma = (\gamma_n)_{n \geq 0}$  be a sequence of positive real numbers, we propose the following algorithm:

**Algorithm 1.** For optimization problem (4.7):

- 1, Let  $\varphi_0 := \phi_{K,M}|_{\mathcal{N}_R}$ , where  $\phi_{K,M}$  is defined in (3.7).
- 2, Given  $\varphi_n$ , compute  $u_{h,T}(\varphi_n)$  and a sub-gradient  $\nabla u_{h,T}(\varphi_n)$ .
- 3, Let  $\varphi_{n+1} := P_{\text{Quad}_{0,h}^{K,M}}[\varphi_n - \gamma_n \nabla u_{h,T}(\varphi_n)]$ .
- 4, Go back to step 2.

In the following, we shall discuss essentially three issues: the computation of sub-gradient  $\nabla u_{h,T}(\varphi)$ , the projection from  $B(\mathcal{N}_R)$  to  $\text{Quad}_{0,h}^{K,M}$  and the convergence of the above gradient projection algorithm.

#### 4.2.1 Computation of sub-gradient

Let us fix  $\varphi \in B(\mathcal{N}_R)$ , we then denote by  $(p^j, \tilde{p}^j)$  the unique solution of the following linear system on  $\mathcal{M}_{T,R}$ :

$$\begin{cases} p^j(t_k, x_i) = -\delta_{i,j}, & (t_k, x_i) \in \partial_T \mathcal{M}_{T,R} \cup \partial_R \mathcal{M}_{T,R}, \\ p^j(t_{k+1}, x_i) - \tilde{p}^j(t_k, x_i) + \frac{1}{2} \Delta t (\theta D^2 \tilde{p}^j(t_k, x_i) + (1-\theta) D^2 p^j(t_{k+1}, x_i)) = 0, \\ p^j(t_k, x_i) = \begin{cases} \tilde{p}^j(t_k, x_i), & \text{if } \lambda_h^{\varphi,T,R}(t_k, x_i) > g^\varphi(t_k, x_i), \\ -e_j(x_i), & \text{otherwise.} \end{cases} & (t_k, x_i) \in \mathring{\mathcal{M}}_{T,R}. \end{cases} \quad (4.14)$$

where  $e_j \in B(\mathcal{N}_R)$  is defined by  $e_j(x_i) := \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$  Let  $p_0^j := p^j(0, \cdot)$ .

**Proposition 4.6.** Let CFL condition (4.3) hold true, then the vector

$$\nabla u_{h,T}(\varphi) := \left( \mu_0(\text{lin}^R[p_0^j]) + \mu_1(\text{lin}^R[e_j]) \right)_{-2m \leq j \leq 2m} \quad (4.15)$$

is a sub-gradient of map  $\varphi \mapsto u_{h,T}(\varphi)$ .

**Proof.** Let us first consider the Markov chain  $\Lambda$  introduced in the proof of Proposition 4.4. By (4.13), we have for every perturbation  $\Delta\varphi \in B(\mathcal{N}_R)$ ,

$$u_{h,T}(\varphi + \Delta\varphi) = \bar{u}_{h,T}^{\tau_h(\varphi + \Delta\varphi)}(\varphi + \Delta\varphi) \geq \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi + \Delta\varphi).$$

It follows still by (4.13) that

$$u_{h,T}(\varphi + \Delta\varphi) - u_{h,T}(\varphi) \geq \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi + \Delta\varphi) - \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi),$$

which implies that

$$\left( \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi + e_j) - \bar{u}_{h,T}^{\tau_h(\varphi)}(\varphi) \right)_{-r \leq j \leq r} \quad (4.16)$$

is a sub-gradient of  $u_{h,T}$  at  $\varphi$  since  $\psi \mapsto \bar{u}_{h,T}^{\tau(\varphi)}(\psi)$  is linear by its definition in (4.11). Finally, by the definition of  $\tau_h(\varphi)$  in (4.12) as well as (4.2) and (4.14), it follows that

$$p^j(t_k, x_i) = - \mathbb{E} [ e_j(\Lambda_{\tau_h(\varphi)}) \mid \Lambda_{t_k} = x_i ].$$

And hence the sub-gradient (4.16) coincides with  $\nabla u_{h,T}(\varphi)$  defined in (4.15).  $\square$

#### 4.2.2 Projection

To compute the projection  $P_{\text{Quad}_{0,h}^{K,M}}$  from  $B(\mathcal{N}_R)$  to  $\text{Quad}_{0,h}^{K,M}$ , we still need to specify the norm equipped on  $B(\mathcal{N}_R)$ . In order to make the projection algorithm simple, we shall introduce an invertible linear map from  $B(\mathcal{N}_R)$  to  $\mathbb{R}^{2r+1}$ , then equip on  $B(\mathcal{N}_R)$  the norm induced by the classical  $L^2$ -norm on  $\mathbb{R}^{2r+1}$ .

Let  $\mathcal{L}_R : B(\mathcal{N}_R) \rightarrow \mathbb{R}^{2r+1}$  be defined by

$$\xi_i = \begin{cases} \varphi(x_i) - \varphi(x_{i-1}), & \text{for } 0 < i \leq 2m, \\ \varphi(x_0), & \text{for } i = 0, \\ \varphi(x_i) - \varphi(x_{i-1}), & \text{for } -2m \leq i < 0. \end{cases} \quad (4.17)$$

We define the norm  $|\cdot|_R$  on  $B(\mathcal{N}_R)$  (easily be verified) by

$$|\varphi|_R := |\mathcal{L}_R(\varphi)|_{L^2(\mathbb{R}^{2r+1})}, \quad \forall \varphi \in B(\mathcal{N}_R).$$

Denote

$$\begin{aligned} E_0^{K,M} &:= \{ \mathcal{L}_R \varphi : \varphi \in \text{Quad}_0^{K,M} \} \\ &= \left\{ \xi \in \mathbb{R}^{2r+1} : 0 = \xi_0 \leq \xi_{\pm 1} \leq \dots \leq \xi_{\pm 2m} \leq 4KM\Delta x, \right. \\ &\quad \left. \xi_{\pm i} = K(x_{i+1}^2 - x_i^2), \forall 2m < i \leq r \text{ and } \sum_{i=1}^{2m} \xi_i = \sum_{i=-1}^{-2m} \xi_i = 4KM^2 \right\}. \end{aligned}$$

Then the projection  $P_{\text{Quad}_{0,h}^{K,M}}$  from  $B(\mathcal{N}_R)$  to  $\text{Quad}_{0,h}^{K,M}$  under norm  $|\cdot|_R$  is equivalent to the projection from  $\mathbb{R}^{2r+1}$  to  $E_0^{K,M}$ , which consists in solving a quadratic minimization problem :

$$\xi^z = \arg \min_{\xi \in E_0^{K,M}} \sum_{i=-r}^r (z_i - \xi_i)^2, \quad \text{for a given } z \in \mathbb{R}^{2r+1}. \quad (4.18)$$

Clearly, for every  $z \in \mathbb{R}^{2r+1}$ ,  $\xi_0^z = 0$  and the above optimization problem (4.18) can be decomposed into two optimization problems:

$$\min_{\xi \in E_{0,+}^{K,M}} \sum_{i=1}^{2m} (z_i - \xi_i)^2 \quad \text{and} \quad \min_{\xi \in E_{0,-}^{K,M}} \sum_{i=-1}^{-2m} (z_i - \xi_i)^2, \quad (4.19)$$

where

$$E_{0,+}^{K,M} := \left\{ \xi = (\xi_i)_{1 \leq i \leq 2m} : 0 \leq \xi_1 \leq \dots \leq \xi_{2m} \leq 4KM\Delta x, \sum_{i=1}^{2m} \xi_i = 4KM^2 \right\},$$

$$E_{0,-}^{K,M} := \left\{ \xi = (\xi_i)_{-1 \geq i \geq -2m} : 0 \leq \xi_{-1} \leq \dots \leq \xi_{-2m} \leq 4KM\Delta x, \sum_{i=-1}^{-2m} \xi_i = 4KM^2 \right\}.$$

Here in place of optimization problem (4.19), we shall consider a similar but more general optimization problem and give an algorithm for it. Let  $a = (a_i)_{1 \leq i \leq m} \in \mathbb{N}^m$  and  $A \in \mathbb{R}^+$  such that  $0 < A < \sum_{i=1}^m a_i$ , we define

$$\mathcal{K}_m^a := \left\{ \xi = (\xi_i)_{1 \leq i \leq m} \in \mathbb{R}^m : \xi_1 \leq \dots \leq \xi_m \right\},$$

$$\mathcal{K}_m^A := \left\{ \xi = (\xi_i)_{1 \leq i \leq m} \in [0, 1]^m : \sum_{i=1}^m a_i \xi_i = A \right\}, \text{ and } \mathcal{K}_m^{a,A} := \mathcal{K}_m^a \cap \mathcal{K}_m^A.$$

The projection  $P_{\mathcal{K}_m^{a,A}}(z)$  of  $z \in \mathbb{R}^m$  to  $\mathcal{K}_m^{a,A}$  is to solve the optimization problem

$$\xi_m^{a,A,z} := \arg \min_{\xi \in \mathcal{K}_m^{a,A}} \sum_{i=1}^m a_i (z_i - \xi_i)^2. \quad (4.20)$$

Similarly, the projection  $P_{\mathcal{K}_m^a}$  ( resp.  $P_{\mathcal{K}_m^A}$  ) is defined by the optimization problem (4.20), where  $\mathcal{K}_m^{a,A}$  in the formula is replaced by  $\mathcal{K}_m^a$  ( resp.  $\mathcal{K}_m^A$  ), and the projected element  $\xi_m^{a,A,z}$  is replaced by  $\xi_m^{a,z}$  ( resp.  $\xi_m^{A,z}$  ).

In the following, we shall show that

$$P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a},$$

and give the algorithms for both  $P_{\mathcal{K}_m^a}$  and  $P_{\mathcal{K}_m^A}$ . With these algorithms, we can deduce easily an algorithm for the projection  $P_{E_{K,M}^+}$ . We just remark that similar algorithm to compute the convex envelope of a function is discussed in Page 143-145 of Edelsbrunner [8].

Given  $a \in \mathbb{N}^m$  and  $z \in \mathbb{R}^m$ , we define  $S^{a,z} \in \mathbb{R}^{\sum_{i=1}^m a_i}$  by  $S_k^{a,z} := z_j$  for  $\sum_{i=1}^{j-1} a_i < k \leq \sum_{i=1}^j a_i$ , and a function  $F^{a,z}$  defined on the grid  $\mathbb{N} \cap [0, 1 + \sum_{i=1}^m a_i]$  by

$$F^{a,z}(0) := 0 \quad \text{and} \quad F^{a,z}(k) := \sum_{i=1}^k S_i^{a,z}. \quad (4.21)$$

**Lemma 4.7.** *Let  $z \in \mathbb{R}^m$  such that  $z_k \geq z_{k+1}$ , then  $(\xi_m^{a,z})_k = (\xi_m^{a,z})_{k+1}$  and  $(\xi_m^{a,A,z})_k = (\xi_m^{a,A,z})_{k+1}$  for  $\xi_m^{a,z} = P_{\mathcal{K}_m^a}(z)$  and  $\xi_m^{a,A,z} = P_{\mathcal{K}_m^{a,A}}(z)$ . And therefore, in this case, the projections  $P_{\mathcal{K}_m^a}(z)$  and  $P_{\mathcal{K}_m^{a,A}}(z)$  are equivalent to  $P_{\mathcal{K}_{m-1}^{\tilde{a}}}(\tilde{z})$  and  $P_{\mathcal{K}_{m-1}^{\tilde{a},A}}(\tilde{z})$  with*

$$\tilde{a}_i = \begin{cases} a_i, & 1 \leq i \leq k-1, \\ a_k + a_{k+1}, & i = k, \\ a_{i+1}, & k+1 \leq i \leq m-1, \end{cases} \quad \text{and} \quad \tilde{z}_i = \begin{cases} z_i, & 1 \leq i \leq k-1, \\ \frac{a_k z_k + a_{k+1} z_{k+1}}{a_k + a_{k+1}}, & i = k, \\ z_{i+1}, & k+1 \leq i \leq m-1, \end{cases} \quad (4.22)$$

in sense that  $S^{a,\xi_m^{a,z}} = S^{\tilde{a},\xi_{m-1}^{\tilde{a},\tilde{z}}}$  and  $S^{a,\xi_m^{a,A,z}} = S^{\tilde{a},\xi_{m-1}^{\tilde{a},\tilde{z}}}$ , where  $\xi_{m-1}^{\tilde{a},\tilde{z}} = P_{\mathcal{K}_{m-1}^{\tilde{a}}}(\tilde{z})$  and  $\xi_{m-1}^{\tilde{a},A,\tilde{z}} = P_{\mathcal{K}_{m-1}^{\tilde{a},A}}(\tilde{z})$ .

**Proof.** Given  $\xi \in \mathbb{R}^m$  such that  $\xi_{k+1} > \xi_k$ , there is  $\varepsilon > 0$  satisfying that  $\xi_{k+1} = \xi_k + (1 + \frac{a_k}{a_{k+1}})\varepsilon$ .

Let  $\hat{\xi}$  be defined by  $\hat{\xi}_i = \begin{cases} \hat{\xi}_k + \varepsilon, & i = k, k+1, \\ \xi_i, & \text{otherwise,} \end{cases}$  we will show that

$$\sum_{i=1}^m a_i (\hat{\xi}_i - z_i)^2 < \sum_{i=1}^m a_i (\xi_i - z_i)^2. \quad (4.23)$$

Thus such a  $\xi$  is not optimal since  $\xi \in \mathcal{K}_m^a$  ( resp.  $\mathcal{K}_m^{a,A}$  ) implies that  $\hat{\xi} \in \mathcal{K}_m^a$  ( resp.  $\mathcal{K}_m^{a,A}$  ) also. And therefore,  $(\xi_m^{a,z})_k = (\xi_m^{a,z})_{k+1}$  and  $(\xi_m^{a,A,z})_k = (\xi_m^{a,A,z})_{k+1}$ .

Indeed, (4.23) holds since with the above given  $\xi$  and  $\hat{\xi}$ ,

$$\begin{aligned} & \sum_{i=1}^m a_i (\xi_i - z_i)^2 - \sum_{i=1}^m a_i (\hat{\xi}_i - z_i)^2 \\ &= a_k (\xi_k - z_k)^2 + a_{k+1} (\xi_k + (1 + \frac{a_k}{a_{k+1}})\varepsilon - z_{k+1})^2 \\ & \quad - a_k (\xi_k + \varepsilon - z_k)^2 - a_{k+1} (\xi_k + \varepsilon - z_{k+1})^2 \\ &= \frac{a_k}{a_{k+1}} (a_k + a_{k+1}) \varepsilon^2 + 2 a_k \varepsilon (z_k - z_{k+1}) > 0. \end{aligned}$$

Finally, the equivalence between  $P_{\mathcal{K}_m^a}(z)$  ( resp.  $P_{\mathcal{K}_m^{a,A}}(z)$  ) and  $P_{\mathcal{K}_{m-1}^{\tilde{a}}}(\tilde{z})$  ( resp.  $P_{\mathcal{K}_{m-1}^{\tilde{a},A}}(\tilde{z})$  ) is from the fact that for every  $\xi$  such that  $\xi_k = \xi_{k+1}$ ,

$$\sum_{i=1}^m a_i (z_i - \xi_i)^2 = \sum_{i=1}^{m-1} \tilde{a}_i (\tilde{z}_i - \tilde{\xi}_i)^2 + a_k z_k^2 + a_{k+1} z_{k+1}^2 - (a_k + a_{k+1}) \frac{(z_k + z_{k+1})^2}{4},$$

$$\text{where } \tilde{\xi}_i = \begin{cases} \xi_i, & i \leq k-1, \\ \xi_k, & i = k, k+1, \\ \xi_{i-1}, & k+2 \leq i \leq m-1. \end{cases} \quad \square$$

Lemma 4.7 gives an algorithm for projection  $P_{\mathcal{K}_m^a}$  which finishes with less than  $m$  steps. And it simplifies the projection  $P_{\mathcal{K}_m^{a,A}}$ .

**Algorithm 2.** For projection  $P_{\mathcal{K}_m^a}(z)$ :

- 1, Given system parameters  $(m, a, z)$ , stop if  $m = 1$ .
- 2, Find  $k$  such that  $z_k \geq z_{k+1}$ , stop if it does not exist.
- 3, With the found  $k$  in step 2, reduce parameters  $(m, a, z)$  to  $(m-1, \tilde{a}, \tilde{z})$  as in equation (4.22).
- 4, Go to 1.

**Proposition 4.8.**  $P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a}$ , and for every  $z \in \mathbb{R}^m$ ,  $F^{a,\xi}$  (with  $\xi := P_{\mathcal{K}_m^a}(z)$ ) is the convex envelope of  $F^{a,z}$ , where the functions  $F^{a,\xi}$  and  $F^{a,z}$  are define in (4.21)

**Proof.** Suppose that the entrance data of Algorithm 2 is  $(m_1, a_1, z_1)$  and exit data is  $(m_2, a_2, z_2)$ , then clearly  $P_{\mathcal{K}_{m_2}^{a_2}}(z_2) = z_2$ . And by Lemma 4.7, we have  $S^{a_1, \xi_1} = S^{a_2, z_2}$  (with  $\xi_1 := P_{\mathcal{K}_{m_1}^{a_1}}(z_1)$ ) and  $S^{a_1, \xi_1^A} = S^{a_2, \xi_2^A}$  ( with  $\xi_1^A := P_{\mathcal{K}_{m_1}^{a_1,A}}(z_1)$  and  $\xi_2^A := P_{\mathcal{K}_{m_2}^{a_2,A}}(z_2)$  ), from which we deduce that,  $P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a}$ .

To see that  $F^{a,\xi}$  (with  $\xi := P_{\mathcal{K}_m^a}(z)$ ) is the convex envelope of  $F^{a,z}$ , it is enough to verify that at every step in Algorithm 2,  $F^{a,\hat{z}}$  is greater than the convex envelope of  $F^{a,z}$ . And at the exit,  $F^{a,\xi}$  is a convex function.  $\square$

Now, we shall prove that  $P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a}$ , in this order, we just need to show that for every  $z \in \mathcal{K}_m^a$ ,  $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^A}(z)$ . In fact, we shall give an algorithm of projection  $P_{\mathcal{K}_m^A}(z)$  for  $z \in \mathcal{K}_m^a$ , and then verify that  $P_{\mathcal{K}_m^A}(z) \in \mathcal{K}_m^{a,A}$ .

Given  $\nu \in \mathbb{R}$ , let us denote by  $z - \nu$  the sequence  $(z_i - \nu)_{1 \leq i \leq m}$ , and by  $z^\nu$  the sequence  $(z_i^\nu)_{1 \leq i \leq m} = (0 \vee (z_i - \nu) \wedge 1)_{1 \leq i \leq m}$ .

**Lemma 4.9.** *Given  $\nu \in \mathbb{R}$ ,  $z \in \mathbb{R}^m$ , then  $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^{a,A}}(z - \nu)$  and  $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^A}(z - \nu)$ . In addition, if  $z \in \mathcal{K}_m^a$ , then there is  $\hat{\nu} \in \mathbb{R}$  such that  $\sum_{i=1}^m a_i z_i^{\hat{\nu}} = A$  and  $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^{a,A}}(z) = z^{\hat{\nu}}$ . And it follows that  $P_{\mathcal{K}_m^{a,A}} = P_{\mathcal{K}_m^{a,A}} \circ P_{\mathcal{K}_m^a} = P_{\mathcal{K}_m^A} \circ P_{\mathcal{K}_m^a}$ .*

**Proof.** To prove that  $P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^{a,A}}(z - \nu)$  or  $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^A}(z - \nu)$ , it is enough to see that for every  $\xi \in \mathbb{R}^m$  such that  $\sum_{i=1}^m a_i \xi_i = A$ ,

$$\sum_{i=1}^m a_i (z_i - \nu - \xi_i)^2 = \sum_{i=1}^m a_i (z_i - \xi_i)^2 + \nu^2 \sum_{i=1}^m a_i - 2\nu \left( \sum_{i=1}^m a_i z_i - A \right).$$

For the existence of  $\hat{\nu}$ , we remark that  $\nu \mapsto \sum_{i=1}^m a_i z_i^\nu$  is continuous, and that  $0 < A < \sum_{i=1}^m a_i$  is supposed at the beginning of the section. Clearly, by its definition,  $z^\nu$  is the projected element of  $z - \nu$  to  $[0, 1]^m$  in sense that  $\xi_0 = z^\nu$  minimizes  $\sum_{i=1}^m a_i (z_i - \nu - \xi_i)^2$  among all  $\xi \in [0, 1]^m$ . Then for  $z \in \mathcal{K}_m^a$ , it is easy to verify that  $z^{\hat{\nu}} \in \mathcal{K}_m^{a,A} \subset \mathcal{K}_m^A \subset [0, 1]^m$  with the found  $\hat{\nu}$ . Therefore  $P_{\mathcal{K}_m^A}(z) = P_{\mathcal{K}_m^{a,A}}(z) = P_{\mathcal{K}_m^A}(z - \hat{\nu}) = P_{\mathcal{K}_m^{a,A}}(z - \hat{\nu}) = z^{\hat{\nu}}$ .  $\square$

**Algorithm 3.** *To find  $\hat{\nu}$  such that  $\sum_{i=1}^m a_i z_i^{\hat{\nu}} = A$ :*

- 1, Set  $z_0 = -\infty$  and  $z_{m+1} = \infty$ .
- 2, Find the maximum  $k$  such that  $\sum_{i=1}^m a_i z_i^{z_{k-1}} \geq A$  and  $\sum_{i=1}^m a_i z_i^{z_k} \leq A$ , then  $z_{k-1} \leq \hat{\nu} \leq z_k$ .
- 3, Find the minimum  $j$  such that  $\sum_{i=1}^m a_i z_i^{z_{j+1}-1} \leq A$  and  $\sum_{i=1}^m a_i z_i^{z_j-1} \geq A$ , then  $z_j - 1 \leq \hat{\nu} \leq z_{j+1} - 1$ .
- 4, Set  $\hat{\nu} = \frac{\sum_{i=j+1}^m a_i + \sum_{i=k}^j a_i z_i - A}{\sum_{i=k}^j a_i}$  when  $k \leq j$ , and  $\hat{\nu} = z_{k-1}$  when  $k = j + 1$ .

By the way how to find  $k$  and  $j$ , we can easily have  $k \leq j + 1$ , then step 4 of Algorithm 3

gives the right  $\hat{\nu}$  since  $z_i^{\hat{\nu}} = \begin{cases} 0, & \text{if } i \leq k - 1, \\ 1, & \text{if } i \geq j + 1, \\ z_i - \hat{\nu}, & \text{otherwise.} \end{cases}$  for  $k, j$  found in step 2 and 3, and hence for

$k \leq j$ ,

$$\sum_{i=k}^j a_i (z_i - \hat{\nu}) + \sum_{i=j+1}^m a_i = A \implies \hat{\nu} = \frac{\sum_{i=j+1}^m a_i + \sum_{i=k}^j a_i z_i - A}{\sum_{i=k}^j a_i}.$$

Finally, we propose the following algorithm for projection  $P_{\text{Quad}_{0,h}^{K,M}}$ :

**Algorithm 4.** *For projection  $P_{\text{Quad}_{0,h}^{K,M}}$  in (4.18):*



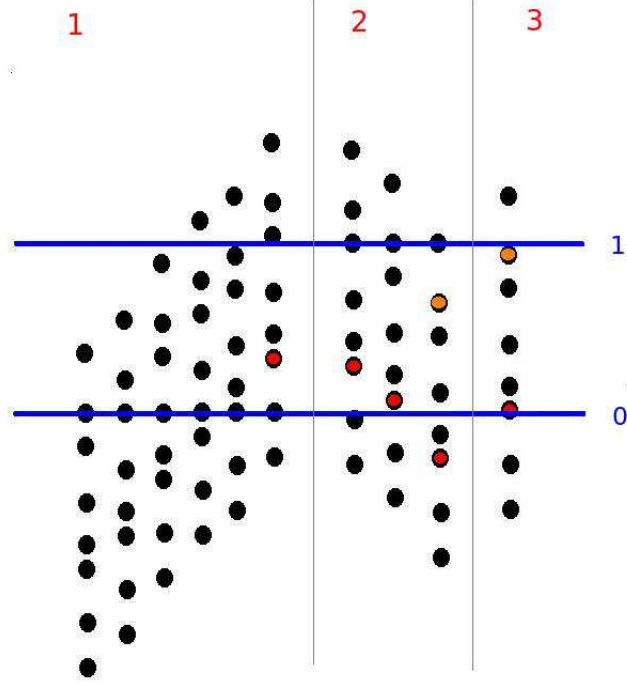


Figure 2: An illustration of Algorithm 3.

- 1, Compute the convex envelope  $\hat{\varphi}$  of  $\varphi$  on  $[0, 2M]$  and on  $[-2M, 0]$ .
- 2, Set  $z = \mathcal{L}_R(\hat{\varphi}|_{\mathcal{N}_R})$ , use Algorithm 3 to compute  $P_{E_0^{K,M}}(u)$ .
- 3, Let  $P_{\text{Quad}_{0,h}^{K,M}}(\varphi) = \mathcal{L}_R^{-1}P_{E_0^{K,M}}(z)$ .

#### 4.2.3 Convergence rate

We shall give a convergence rate for the gradient projection algorithm. In preparation, let us first give a bound for the sub-gradients  $\nabla u_{h,T}$ .

**Proposition 4.10.** *Let  $\varphi_1, \varphi_2 \in B(\mathcal{N}_R)$ , then under the CFL condition (4.3),*

$$|u_{h,T}(\varphi_1) - u_{h,T}(\varphi_2)| \leq 2|\varphi_1 - \varphi_2|_\infty, \quad (4.24)$$

and it follows that

$$|\nabla u_{h,T}(\varphi)|_R \leq 2\sqrt{2m+1} = 2\sqrt{\frac{2M}{\Delta x} + 1}, \quad \forall \varphi \in B(\mathcal{N}_R). \quad (4.25)$$

**Proof.** Under the CFL condition (4.3), the  $\theta$ -scheme is monotone, which implies that  $|\lambda_h^{\varphi,T,R,\varphi_1} - \lambda_h^{\varphi,T,R,\varphi_2}|_\infty \leq |\varphi_1 - \varphi_2|_\infty$ , and hence by the definition of  $u_{h,T}$  in (4.4), (4.24) holds true. Next, denote  $\xi^i := \mathcal{L}_R(\varphi_i)$ ,  $i = 1, 2$ , then by Cauchy-Schwarz inequality,

$$|\varphi_1 - \varphi_2|_\infty \leq \max\left(\sum_{i=0}^{2m} |\xi_i^1 - \xi_i^2|, \sum_{i=0}^{-2m} |\xi_i^1 - \xi_i^2|\right) \leq \sqrt{2m+1} \cdot \|\xi^1 - \xi^2\|_{\mathbb{L}^2},$$

which implies immediately (4.25).  $\square$

Finally, let us finish this section by providing a convergence rate of the proposed gradient projection algorithm. Denote

$$\Phi := \max_{\varphi_1, \varphi_2 \in \text{Quad}_{0,h}^{K,M}} |\varphi_1 - \varphi_2|_R^2 \leq 4m (4KM\Delta x)^2 \leq 64K^2M^3\Delta x,$$

then from Section 5.3.1 of Ben-Tal and Nemirovski[3], we have the convergence rate:

$$\begin{aligned} \min_{n \leq N} u_{h,T}(\varphi_n) - U_h^{K,M,T} &\leq \frac{\Phi + \sum_{i=n}^N \gamma_n^2 |\nabla u_{h,T}(\varphi_n)|_R^2}{2 \sum_{n=1}^N \gamma_n} \\ &= \frac{32K^2M^3\Delta x + (4\frac{M}{\Delta x} + 2) \sum_{i=n}^N \gamma_n^2}{\sum_{n=1}^N \gamma_n}. \end{aligned} \quad (4.26)$$

For the sequence  $\gamma = (\gamma_n)_{n \geq 1}$ , there are several choices:

- Divergent Series :  $\gamma_n \geq 0$ ,  $\sum_{n=1}^{\infty} \gamma_n = +\infty$  and  $\sum_{n=1}^{\infty} \gamma_n^2 < +\infty$ . We get convergence as  $N \rightarrow \infty$ .
- Optimal stepsizes :  $\gamma_n = \frac{\sqrt{\Phi}}{|\nabla u_{h,T}(\varphi_n)|_R \sqrt{n}}$ , we have by [3] that

$$\min_{n \leq N} u_{h,T}(\varphi_n) - U_h^{K,M,T} \leq O(1) \cdot \frac{16KM\sqrt{2M^2 + M\Delta x}}{\sqrt{N}}.$$

## 5 Numerical example

As shown in Corollary 3.9, the model-free price upper bound of variance swap is  $C_0$  defined in (2.2). Let  $(S_t)_{t \geq 0}$  follow the Black-Scholes dynamics  $dS_t = \sigma S_t dW_t$ , where  $(W_t)_{t \geq 0}$  is a standard Brownian motion, and  $\mu_0 \sim S_{\frac{1}{2}}$  and  $\mu_1 \sim S_1$ . Then

$$C_0 = \mathbb{E} ( S_1^2 - S_{\frac{1}{2}}^2 ) = \mathbb{E} \int_{\frac{1}{2}}^1 \sigma^2 S_t^2 dt = \frac{1}{2} \sigma^2 S_0^2.$$

We set  $\sigma = 0.2$ ,  $S_0 = 1$ , it follows that  $C_0 = 0.02$ . In our implemented example, with a 2.40GHz CPU computer, it takes 57.24 seconds to finish  $4 \times 10^4$  iterations, and we get the numerical upper bound 0.2019, i.e. the relative error is less than 1 %, see also Figure 3.

## 6 Appendix

We give a proof for Theorem 2.5, where we use the weak dynamic programming technique proposed in Bouchard and Touzi [4].

**Proof of Theorem 2.5.** We first introduce

$$\bar{U}^K := \inf_{\phi \in \text{Quad}_0^K} \bar{u}(\phi) \quad \text{and} \quad \bar{U}^{K,M} := \inf_{\phi \in \text{Quad}_0^{K,M}} \bar{u}(\phi),$$

and we claim that

$$\bar{u}(\phi) = u(\phi), \quad \forall \phi \in \text{Quad}_0^{K,M}, \quad (6.1)$$

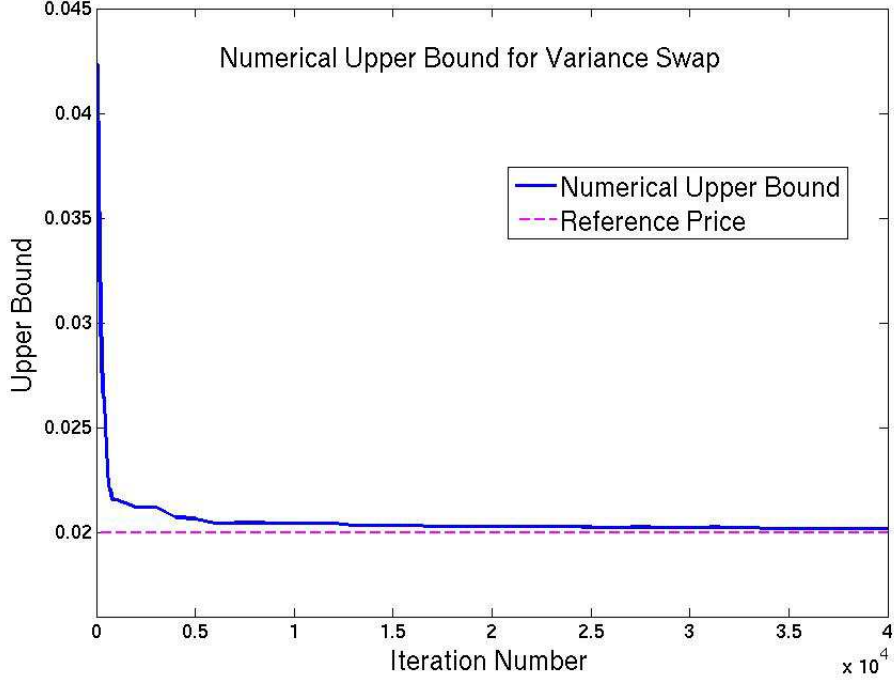


Figure 3: Numerical result for variance swap with approximation parameters:  $T = 0.1$ ,  $K = 1$ ,  $M = 1$ ,  $R = 2$ ,  $\Delta t = 0.002$ ,  $\Delta x = 0.1$  and  $\gamma_n = \sqrt{n}$ .

which implies that  $\bar{U}^{K,M} = U^{K,M}$ . Clearly, by the same arguments as in (3.3) and Proposition 3.2, we have  $\bar{U}^K \rightarrow \bar{U}$  and  $\bar{U}^{K,M} \rightarrow \bar{U}^K$  as  $(K, M) \rightarrow \infty$ . It follows that  $\bar{U} = U$ .

Therefore, it is enough to prove (6.1) to conclude, which is in fact a dynamic programming principle for  $\bar{u}$  defined in (1.5). Moreover, by the dominated convergence theorem,  $\lambda^{\phi, \tau_R}$  defined in (3.8) is a continuous function for every  $\phi \in \text{Quad}$ . Hence  $\lambda^\phi$  is continuous for every  $\phi \in \text{Quad}_0^{K,M}$  by Lemma 3.3. Therefore, it is enough to derive a weak dynamic programming principle following Bouchard and Touzi [4].

Let  $\phi \in \text{Quad}_0^{K,M}$ ,  $\tau \in \mathcal{T}$  which is defined in (1.6), since the stopping time  $\tau$  conditioned on  $W_0$  belongs to  $\mathcal{T}^\infty$ , then by a simple conditioning argument,  $\mathbb{E}[g^\phi(\tau, W_\tau)] \geq \mu_0(\lambda_0^\phi)$ , which implies that  $u(\phi) \leq \bar{u}(\phi)$ . On the other hand, as in the proof of Theorem 4.1 in [4], for every  $\varepsilon > 0$ , there is a countable subdivision  $\Delta = (\Delta_n)_{n \geq 1}$  of  $\mathbb{R}$ , a sequence of stopping times  $(\tau_n^\varepsilon)_{n \geq 1}$  in  $\mathcal{T}^\infty$  such that  $\mathbb{E}[g^\phi(\tau_n^\varepsilon, x + B_{\tau_n^\varepsilon})] \leq \lambda_0^\phi(x) + \varepsilon$ ,  $\forall x \in \Delta_n$ . We then construct  $\tau^\varepsilon \in \mathcal{T}$  by  $\tau^\varepsilon(W) := \sum_{n=1}^{\infty} \tau_n^\varepsilon(W - W_0) \mathbf{1}_{W_0 \in \Delta_n}$ , so that  $\mathbb{E}[g^\phi(\tau^\varepsilon, W_{\tau^\varepsilon})] \leq \mu_0(\lambda_0^\phi) + \varepsilon$ . By the arbitrariness of  $\varepsilon > 0$ , we then get  $\bar{u}(\phi) \leq \mu_0(\lambda_0^\phi) + \mu_1(\phi) = u(\phi)$ , and hence establish (6.1) which concludes the proof of Proposition.  $\square$

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