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# Facial parity edge colouring

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## Abstract

The *facial parity edge colouring* of a connected bridgeless plane graph is such an edge colouring in which no two face-adjacent edges (consecutive edges of a facial walk of some face) receive the same colour, in addition, for each face  $\alpha$  and each colour  $c$ , either no edge or an odd number of edges incident with  $\alpha$  is coloured with  $c$ . From the Vizing's theorem it follows that every 3-connected plane graph has a such colouring with at most  $\Delta^* + 1$  colours, where  $\Delta^*$  is the size of the largest face. In this paper we prove that any connected bridgeless plane graph has a facial parity edge colouring with at most 92 colours.

*Keywords:* plane graph, facial walk, edge colouring

*Math. Subj. Class.:* 05C10, 05C15

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## 1 Introduction

One of the motivations for this paper has come from recent papers of Bunde et al. [1, 2] who introduced parity edge colourings of simple graphs. Studying the parity of the usage of colours along walks suggested two edge colouring parameters that have interesting properties and applications. A *parity walk* in an edge colouring of a simple graph is a walk along which each colour is used an even number of times. Bunde et al. [2] introduced two parameters. Let  $p(G)$  be the minimum number of colours in an edge colouring of  $G$  having no parity path (a *parity edge colouring*). Let  $\widehat{p}(G)$  be the minimum number of colours in an edge colouring of  $G$  in which every parity walk is closed (a *strong parity edge colouring*). Since incident edges of the same colour would form a parity path of length 2, every parity edge colouring is a proper edge colouring, and hence  $p(G) \geq \chi'(G)$ , where  $\chi'(G)$  is the chromatic index of  $G$ . Since a path is an open walk, no strong parity edge colouring has a parity path. Hence, every strong parity edge colouring is a parity edge colouring and  $\widehat{p}(G) \geq p(G)$  for every graph  $G$ . Although there are graphs  $G$  with  $\widehat{p}(G) > p(G)$  [2], it remains unknown how large  $\widehat{p}(G)$  can be when  $p(G) = k$ . Elementary results on these parameters appear in [2]. In [1] it is proved that  $\widehat{p}(K_n) = 2^{\lceil \log(n) \rceil} - 1$  for all  $n$ . Moreover, the optimal strong parity edge colouring of the complete  $n$ -vertex graph  $K_n$  is unique when  $n$  is a power of 2. The authors of [2] mentioned that computing  $p(G)$  or  $\widehat{p}(G)$  is NP-hard even when  $G$  is a tree. Clearly, the parity edge colouring is such a colouring that each path uses at least one colour an odd number of times.

The vertex version of this problem (*strong parity vertex colouring*) with some restrictions was introduced in [3]. The authors of [3] conjectured that there is a constant  $K$  such that the vertices of any 2-connected plane graph can be coloured with at most  $K$  colours in such a way that for each face  $\alpha$  and each colour  $c$ , either no vertex or an odd number of vertices incident with  $\alpha$  is coloured with  $c$ .

Another motivation for this work has come from the papers of Pyber [5] and Mátrai [4]. A graph is called *odd* if the degree of its vertices is odd or zero. Pyber raises the problem of edge covering with odd subgraphs in [5] as the counterpart of even subgraph covering problems. He proved that the edges of every finite simple graph can be covered by at most 4 edge-disjoint odd subgraphs; moreover, if the number of vertices is even then 3 odd subgraphs are sufficient. For not necessarily edge-disjoint coverings we have the following question: Is it true that every graph can be covered by at most 3 odd subgraphs? Mátrai in [4] showed that every finite simple graph can be covered by 3 odd subgraphs and he found an infinite sequence of finite simple connected graphs not coverable by 3 edge-disjoint odd subgraphs.

Pyber's result implies the following: The edges of any 3-connected plane graph  $G$  can be coloured by at most 4 colours in such a way that for each face  $\alpha$  and each colour  $c$ , either no edge or an odd number of edges incident with  $\alpha$  is coloured with  $c$ . It is sufficient to consider the dual  $G^*$  of  $G$  and its edge cover with at most 4 edge-disjoint odd subgraphs. This cover induces the required colouring of  $G$ .

If we add a requirement that such a colouring must be proper then it is not clear whether there exists such a colouring with  $K$  colours, where  $K$  is an absolute constant. From Vizing's theorem [6] it follows that every 3-connected plane graph  $G$  has such a colouring which uses at most  $\Delta^* + 1$  colours, where  $\Delta^*$  is the size of the largest face. Consider a proper edge colouring of the dual graph  $G^*$ . This colouring induces a colouring of  $G$  in a natural way. It is such a colouring in which, for each face  $\alpha$  of  $G$ , all the edges in the boundary of  $\alpha$  have distinct colours.

In this paper we show that each connected bridgeless plane graph has a facial parity edge colouring using at most 92 colours.

## 2 Notation

Let us introduce the notation used in this paper. A graph which can be embedded in the plane is called *planar graph*; a fixed embedding of a planar graph is called *plane graph*.

A *bridge* is an edge whose removal increases the number of components. A graph which contains no bridge is said to be *bridgeless*. In this paper we consider connected bridgeless plane graphs, multiple edges and loops are allowed.

Let  $G = (V, E, F)$  be a connected plane graph with the vertex set  $V$ , the edge set  $E$ , and the face set  $F$ . The *degree* of a vertex  $v$ , denoted by  $\deg(v)$ , is the number of edges incident with  $v$ , each loop counting as two edges. A  $k$ -vertex is a vertex of degree  $k$ . For a face  $\alpha$ , the *size* of  $\alpha$ ,  $\deg(\alpha)$ , is defined to be the length of its *facial walk*, i.e. the shortest closed walk containing all edges from the boundary of  $\alpha$ . We often say  $k$ -face for a face of size  $k$ .

Given a graph  $G$  and one of its edges  $e = uv$  (the vertices  $u$  and  $v$  do not have to be different), the *contraction* of  $e$  denoted by  $G/e$  consists of replacing  $u$  and  $v$  by a new vertex adjacent to all the former neighbours of  $u$  and  $v$ , and removing the loop corresponding to the edge  $e$ . (We keep multiple edges if they arise.) Analogously we define the contraction of the set of edges  $H = \{e_1, \dots, e_k\}$  and we denote it by  $G/\{e_1, \dots, e_k\}$  or  $G/H$ .

Two faces are *adjacent* if they share an edge. Two (distinct) edges are *face-adjacent* if they are consecutive edges of a facial walk of some face  $\alpha$ .

A  $k$ -*edge colouring* of a graph  $G$  is a mapping  $\varphi : E(G) \rightarrow \{1, \dots, k\}$ . The *facial parity edge (FPE) colouring* of a connected bridgeless plane graph is such an edge colouring that no two face-adjacent edges receive the same colour, in addition, for each face  $\alpha$  and each colour  $c$ , either no edge or an odd number of edges incident with  $\alpha$  is coloured with  $c$ .

**Question 2.1.** What is the minimum number of colours  $\chi'_{fp}(G)$  that a connected bridgeless plane graph  $G$  has a facial parity edge colouring with at most  $\chi'_{fp}(G)$  colours?

The number  $\chi'_{fp}(G)$  is called the *facial parity chromatic index* of  $G$ .

## 3 Results

**Theorem 3.1.** *Let  $G$  be a connected bridgeless plane graph. Then*

$$\chi'_{fp}(G) \leq 92.$$

The proof uses the method of discharging. Let  $G$  be a counterexample with minimal number of edges, then minimal number of 1-faces, and then minimal number of 2-faces. If  $G$  is a single cycle of length  $d$ ,  $d \leq 5$ , we use exactly  $d$  colours. We consider this to be the first step of induction and call this case trivial.

First, we prove several structural properties of  $G$ .

We say that a face  $\alpha$  is *small* if  $1 \leq \deg(\alpha) \leq 44$  and a face  $\beta$  is *big* if  $\deg(\beta) \geq 45$ .

### 3.1 Reducible configurations

We find such (forbidden) subgraphs  $H$  of  $G$  that the facial parity edge colouring of  $G/H$  using at most 92 colours can be extended to a required colouring of  $G$  using at most 92

colours, which is a contradiction to  $G$  being a counterexample. In the sequel, whenever we speak about an FPE colouring, we always mean an FPE colouring using at most 92 colours.

### 3.1.1 1-faces

**Claim 3.2.** *Each vertex of  $G$  is incident with at most one 1-face.*

*Proof.* Let  $v$  be a vertex incident with at least two 1-faces  $\alpha_1$  and  $\alpha_2$ . If we split  $v$  into two vertices  $v_1$  and  $v_2$  in such a way that  $\alpha_1$  and  $\alpha_2$  become a 2-face  $\alpha$  and face-adjacency of all edges incident with  $v$  is preserved, we obtain a graph  $G'$ , see Figure 1. It has the same number of edges as  $G$ , but fewer 1-faces. Thus, it is not a counterexample and we can find an FPE colouring  $\varphi'$  of  $G'$ . It induces an edge colouring  $\varphi$  of  $G$  in a natural way. It is easy to see that  $\varphi$  is an FPE colouring.  $\square$



Figure 1: A vertex incident with at least two 1-faces can be split into two vertices, reducing the number of 1-faces. Similarly, one can reduce a 1-face and a  $d$ -face ( $2 \leq d \leq 4$ ) incident with the same vertex.

**Claim 3.3.** *Each vertex of  $G$  incident with a 1-face is not incident with any other  $d$ -face for  $2 \leq d \leq 4$ .*

*Proof.* We use the same reduction as in the proof of Claim 3.2. We split the vertex  $v$  incident with a 1-face  $\gamma$  and a  $d$ -face  $\alpha$  ( $2 \leq d \leq 4$ ) in such a way that the faces  $\alpha$  and  $\gamma$  become a  $(d + 1)$ -face  $\alpha'$  and face-adjacency of all edges incident with  $v$  is preserved, see Figure 1 for illustration. Let the reduced graph be  $G'$ . It has fewer 1-faces than  $G$ , therefore, it has an FPE colouring  $\varphi'$ . The face  $\alpha'$  has size at most five, therefore, its edges are coloured using  $d + 1$  different colours. Thus, the colouring  $\varphi$  of  $G$  induced by the colouring  $\varphi'$  of  $G'$  is an FPE colouring, too.  $\square$

**Claim 3.4.** *Each vertex of  $G$  incident with a 2-face is not incident with any  $d$ -face for  $2 \leq d \leq 3$ .*

*Proof.* We use the same reduction as in the proofs of Claims 3.2 and 3.3. We omit the details.  $\square$

### 3.1.2 Small faces

**Claim 3.5.** *There are no two adjacent small faces in  $G$ .*

*Proof.* Let  $\alpha_1$  and  $\alpha_2$  be two adjacent small faces in  $G$ .

If both  $\alpha_1$  and  $\alpha_2$  are 1-faces, the graph consists of a single vertex and a loop; it has an FPE colouring using 1 colour.

Let  $\alpha_1$  be a 1-face and  $\alpha_2$  be a  $d$ -face,  $d \geq 2$ ; let  $e$  be the loop they share, see Figure 2. Then the graph  $G' = G/e$  has fewer edges than  $G$ , therefore, it has an FPE colouring  $\varphi'$ . Let  $\alpha'$  be a face in  $G'$  corresponding to  $\alpha_1$  and  $\alpha_2$  in  $G$ . Since  $\alpha_2$  is a small face, at most 43 colours occur on the edges incident with  $\alpha'$ . To extend the colouring  $\varphi'$  of  $G'$  to an FPE colouring of  $G$ , it suffices to colour the edge  $e$  with any colour that does not occur on  $\alpha'$ .

Let  $\alpha_1$  and  $\alpha_2$  be two small faces of size at least 2 and let  $e$  be the edge they share, see Figure 2. The graph  $G' = G/e$  has fewer edges than  $G$ , therefore, it has an FPE colouring  $\varphi'$ . Let  $\alpha'_1$  and  $\alpha'_2$  be the faces of  $G'$  corresponding to the faces  $\alpha_1$  and  $\alpha_2$  in  $G$ . (Since  $\alpha_1$  and  $\alpha_2$  are faces of size at least 2, the size of  $\alpha'_1$  and  $\alpha'_2$  is at least 1.)

Consider the set of colours different from the colours occurring on the edges of  $\alpha'_1$  and  $\alpha'_2$  (the colours admissible for the edge  $e$ ). Since  $\alpha_1$  and  $\alpha_2$  are small, at most  $2 \cdot (44 - 1) = 86$  colours occur on the edges incident with  $\alpha'_1$  and  $\alpha'_2$ . Hence, there is an admissible colour  $c$ . We can extend  $\varphi'$  to an FPE colouring  $\varphi$  of  $G$  by setting  $\varphi(e) = c$ .  $\square$



Figure 2: Two adjacent small faces form a reducible configuration: one can contract the edge they share.

**Claim 3.6.** Let  $\beta$  be a big face adjacent to two small faces  $\alpha_1$  and  $\alpha_2$ . Let  $e_i$  be an edge incident with  $\beta$  and  $\alpha_i$ ,  $i = 1, 2$ . Then  $e_1$  and  $e_2$  are face-adjacent.

*Proof.* Let  $e_1$  and  $e_2$  not be face-adjacent. See Figure 3 for illustration. The graph  $G' = G/\{e_1, e_2\}$  has fewer edges than  $G$ , therefore, it has an FPE colouring  $\varphi'$ . Let  $\alpha'_1, \alpha'_2, \beta'$  be the faces of  $G'$  corresponding to the faces  $\alpha_1, \alpha_2, \beta$  in  $G$ , respectively.

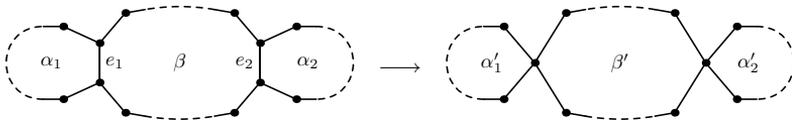


Figure 3: A big face  $\beta$  adjacent to two different small faces  $\alpha_1$  and  $\alpha_2$  forms a reducible configuration unless the edges  $e_1$  and  $e_2$  are face-adjacent.

We extend the colouring  $\varphi'$  of  $G'$  to an FPE colouring of  $G$  in the following way: Consider the set of colours different from the colours occurring on the edges of  $\alpha'_1$  and  $\alpha'_2$ ; also different from the colours occurring on the edges of  $G'$  corresponding to the edges of  $G$  face-adjacent to  $e_1$  and  $e_2$ . There are at least  $92 - 2 \cdot (44 - 1) - 4 = 2$  such colours, say  $c_1$  and  $c_2$ . If at least one of them, say  $c_i$ , already occurs on  $\beta'$ , we set  $\varphi(e_1) = \varphi(e_2) = c_i$ . If none of them occurs on  $\beta'$ , we set  $\varphi(e_i) = c_i$ ,  $i = 1, 2$ .  $\square$

**Claim 3.7.** Each big face is adjacent to at most one 1-face.

*Proof.* It follows from Claims 3.2 and 3.6.  $\square$

**Claim 3.8.** *Each big face is adjacent to at most two small faces.*

*Proof.* Let a big face  $\beta$  be adjacent to small faces  $\alpha_1, \alpha_2,$  and  $\alpha_3$ . Consider the edges that  $\beta$  shares with  $\alpha_i, i = 1, 2, 3$ . It is easy to see that there must be a pair of edges  $e_i$  and  $e_j,$  incident to  $\alpha_i$  and  $\alpha_j,$  respectively ( $i \neq j$ ), which are not face-adjacent. It is a contradiction with Claim 3.6.  $\square$

**3.1.3 Chains of 2-vertices**

**Claim 3.9.** *There is no chain consisting of at least 5 consecutive 2-vertices in  $G$ .*

*Proof.* Let  $v_0e_0v_1e_1 \dots v_pe_pv_{p+1}$  be a chain consisting of  $p$  vertices  $v_1, \dots, v_p$  of degree 2, where  $p \geq 5$ . The graph  $G' = G/\{e_1, e_2, e_3, e_4\}$  has an FPE colouring  $\varphi'$ . Let  $e'_0$  and  $e'_5$  be the edges in  $G'$  corresponding to the edges  $e_0$  and  $e_5$  in  $G$ . Let  $\varphi'(e'_0) = c_1$  and  $\varphi'(e'_5) = c_2$ . The FPE colouring  $\varphi'$  of  $G'$  can be extended to an FPE colouring  $\varphi$  of  $G$  by setting  $\varphi(e_2) = \varphi(e_4) = c_1$  and  $\varphi(e_1) = \varphi(e_3) = c_2$ , see Figure 4.  $\square$

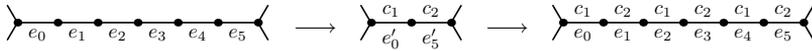


Figure 4: A chain of (at least) five 2-vertices is a reducible configuration.

A  $d$ -face  $\alpha$  is *hanging* on a vertex  $v$ , if all vertices incident with  $\alpha$  are 2-vertices except for the vertex  $v$ . By Claim 3.9 we have  $d \leq 5$ . (If  $\deg(v) = 2$ , the graph  $G$  consists of a single cycle of length at most 5, which is the trivial case).

We colour the vertices of  $G$  with black, blue and white colour in the following way:

Let all 2-vertices be black, all 3-vertices be blue and all  $k$ -vertices for  $k \geq 6$  be white.

A 4-vertex  $v$  is black if there is a face hanging on it, else it is white. See Figure 5 for illustration of all types of black vertices.

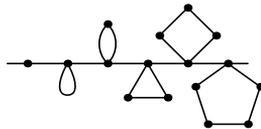


Figure 5: A vertex is black, if it is a 2-vertex, or a 4-vertex with a hanging face.

A 5-vertex  $v$  is blue, if there is a face hanging on it, else it is white.

Observe that any 2-vertex  $v$  is incident with two faces. We say  $v$  is *bad* for both faces it is incident with. Any black 4-vertex  $v$  is incident with a small face  $\alpha$  of size at most 5 and two other faces. The face  $\beta$  adjacent to  $\alpha$  must be big (see Claim 3.5); the vertex  $v$  occurs twice on the facial walk of  $\beta$ . The other face  $\gamma$  can be big or small. We say  $v$  is *bad* for the face  $\gamma$ .

**Claim 3.10.** *For any face  $\alpha$ , there is no chain of at least 5 consecutive vertices bad for  $\alpha$ . Each chain of bad vertices contains at most one bad 4-vertex.*

*Proof.* Let  $v_0e_0v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6$  be a subpath of the facial walk of  $\alpha$  containing 5 vertices  $v_1, \dots, v_5$  bad for  $\alpha$ . It is easy to see that all the edges  $e_0, \dots, e_5$  are incident with the same face  $\beta$  ( $\beta \neq \alpha$ ). If at least two of  $v_1, \dots, v_5$  are bad 4-vertices, we come to a contradiction with Claim 3.5 or with Claim 3.6. If none of them is a 4-vertex, we are in the case of Claim 3.9. If precisely one of them is a 4-vertex, say  $v_i$ , it is incident with a small face  $\gamma$  of size  $d \leq 5$ .

The graph  $G' = G/\{e_1, e_2, e_3, e_4\}$  has an FPE colouring  $\varphi'$ . Let  $e'_0$  and  $e'_5$  be the edges in  $G'$  corresponding to the edges  $e_0$  and  $e_5$  in  $G$ ; let  $\alpha', \beta', \gamma'$  be the faces in  $G'$  corresponding to the faces  $\alpha, \beta, \gamma$  in  $G$ ; let  $v'$  be the vertex in  $G'$  corresponding to  $v_1, \dots, v_5$  in  $G$ . Let  $\varphi'(e'_0) = c_1$  and  $\varphi'(e'_5) = c_2$ . Since  $e'_0$  and  $e'_5$  are face-adjacent in  $G'$ ,  $c_1 \neq c_2$ . To extend  $\varphi'$  to an FPE colouring of  $G$ , we proceed in the following way:

If  $i = 1, 3$ , or  $5$ , we simply set  $\varphi(e_2) = \varphi(e_4) = c_1$  and  $\varphi(e_1) = \varphi(e_3) = c_2$ .

If  $i = 2$  or  $i = 4$ , we set  $\varphi(e_2) = \varphi(e_4) = c_1$  and  $\varphi(e_1) = \varphi(e_3) = c_2$  and switch the order of colours of edges incident with  $\gamma'$ , see Figure 6 for illustration (here the small face  $\gamma$  is a bigon).

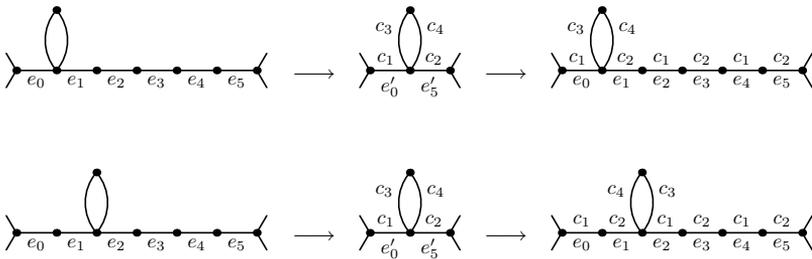


Figure 6: A chain of (at least) five bad black vertices is a reducible configuration as well. □

**Claim 3.11.** *Let  $v$  be a black vertex bad for a small face  $\alpha$ . If  $v$  is a 4-vertex, then the face hanging on it is a 1-face.*

*Proof.* Let  $v$  be a black 4-vertex, let  $\gamma$  be the face hanging on  $v$  of size at least 2 and let  $e_1$  and  $e_2$  be the edges of  $\alpha$  incident with  $v$ . It is easy to see that the edges  $e_1$  and  $e_2$  are incident with the same big face, say  $\beta$ . See Figure 7 for illustration.

There is an edge  $e_\gamma$  incident with  $\gamma$  and  $\beta$ , which is not face-adjacent to  $e_\alpha \in \{e_1, e_2\}$ , which is a contradiction with Claim 3.6. □



Figure 7: Reducible pairs of edges ( $e_\gamma$  and  $e_\alpha$ ). For details see Claims 3.11 and 3.12.

**Claim 3.12.** *Let  $\alpha$  be a small face sharing at least two bad black vertices with a big face  $\beta$ . Then all the bad black vertices incident with  $\alpha$  and  $\beta$  are 2-vertices.*

*Proof.* Let  $v_1$  and  $v_2$  be black vertices incident with  $\alpha$  and  $\beta$ . If  $v_1$  is a 4-vertex, it is bad for  $\alpha$ , therefore, there is a small face  $\gamma$  hanging on  $v_1$ , adjacent to  $\beta$ . By Claim 3.11, the face  $\gamma$  is a 1-face. Thus, we can find an edge (a loop)  $e_\gamma$  incident with  $\gamma$  and  $\beta$  and an edge  $e_\alpha$  incident with  $\alpha$  and  $\beta$ , which are not face-adjacent. This is a contradiction with Claim 3.6. See Figure 7 for illustration.  $\square$

**Claim 3.13.** *Let  $v$  be a black vertex bad for a small  $d$ -face  $\alpha$ ,  $d \in \{2, 3, 4\}$ . Then  $v$  is a 2-vertex.*

*Proof.* Let  $v$  be a black 4-vertex, let  $\gamma$  be the face hanging on  $v$ . By Claim 3.11, the face  $\gamma$  is a 1-face. On the other hand, by Claim 3.3, the face  $\gamma$  cannot be a 1-face, which is a contradiction.  $\square$

**Claim 3.14.** *If a big face  $\beta$  shares a 2-vertex  $v$  with a small face  $\alpha$ , then  $\beta$  is not adjacent to any other small face.*

*Proof.* It follows immediately from Claim 3.6.  $\square$

**Claim 3.15.** *Let  $\gamma$  be a  $d$ -face,  $d \in \{2, 3, 4, 5\}$ , hanging on a vertex  $v$ , adjacent to a big face  $\beta$ . Then  $\beta$  is not adjacent to any other small face.*

*Proof.* It follows immediately from Claim 3.14.  $\square$

### 3.2 Discharging rules

If  $G$  is a minimal counterexample then it contains no reducible configuration. Let the initial charge of each vertex be  $\psi(v) = \deg(v) - 6$  and the initial charge of each face be  $\psi(\alpha) = 2 \deg(\alpha) - 6$ . From Euler's formula we can easily derive that

$$\sum_{\alpha \in F} (2 \deg(\alpha) - 6) + \sum_{v \in V} (\deg(v) - 6) = -12.$$

It is obvious that all the negative charge is in the vertices of degree 2, 3, 4, and 5 and in the faces of size 1 and 2.

**Rule 1:** Let  $\beta$  be a big face.

- If  $\beta$  is adjacent to a single small face  $\alpha$ , it sends 3 units of charge to  $\alpha$ .
- If  $\beta$  is adjacent to two small faces  $\alpha_1$  and  $\alpha_2$ , such that  $\deg(\alpha_1) \leq \deg(\alpha_2)$ , it sends 2 units of charge to  $\alpha_1$  and 1 unit of charge to  $\alpha_2$ . (If  $\deg(\alpha_1) = \deg(\alpha_2)$ , it is decided arbitrarily.)

**Rule 2:** Let  $\beta$  be a big face.

- It sends 2 units of charge to any black vertex bad for  $\beta$ .
- It sends 1 unit of charge to any other black, blue, or white vertex incident with  $\beta$ . (Multiply incident vertices are considered as different.)

**Rule 3:** Let  $\alpha$  be a small face.

- It sends 2 units of charge to any black vertex bad for  $\alpha$ .
- It sends 1 unit of charge to any other black or blue vertex incident with  $\alpha$ .

**Rule 4:** Let  $v$  be a black 4-vertex.

- It sends 2 units of charge to the incident small hanging face  $\gamma$ .

**Rule 5:** Let  $v$  be a blue 5-vertex.

- It sends 2 units of charge to the incident small hanging face  $\gamma$ .

**Rule 6:** Let  $v$  be a  $k$ -vertex,  $k \geq 6$ .

- It sends 2 units of charge to any incident small hanging face  $\gamma$ .

### 3.3 Analysis of the discharging process

#### 3.3.1 Vertices

Every 2-vertex is black and bad for both faces incident with it, hence it receives 2 units of charge from both incident faces (Rules 2 and 3). Its new charge is  $-4 + 2 + 2 = 0$ .

Every 3-vertex is blue, hence it receives 1 unit of charge from all the three incident faces (Rules 2 and 3). Its new charge is  $-3 + 3 \cdot 1 = 0$ .

Every black 4-vertex  $v$  receives 2 units of charge from the face it is bad for (Rules 2 and 3) and  $2 \cdot 1$  units of charge from the doubly-incident big face  $\beta$  (Rule 2). It sends 2 units of charge to the hanging face  $\gamma$  (Rule 4). The new charge of  $v$  is  $-2 + 2 + 2 \cdot 1 - 2 = 0$ .

Every white 4-vertex is incident with at least 2 big faces (see Claim 3.5), therefore, its new charge is at least  $-2 + 2 \cdot 1 = 0$ .

Every blue 5-vertex  $v$  is incident with a hanging face  $\gamma$ , doubly-incident with a big face  $\beta$  and incident with two more faces  $\alpha_1$  and  $\alpha_2$ . Therefore,  $v$  receives  $4 \cdot 1$  units of charge from the incident faces (Rules 2 and 3). It sends 2 units of charge to  $\gamma$  (Rule 5). Therefore, the new charge of  $v$  is  $-1 + 4 \cdot 1 - 2 = 1$ .

Every white 5-vertex is incident with at least 3 big faces (see Claim 3.5), therefore, its new charge is at least  $-1 + 3 \cdot 1 = 2$ .

Every (white)  $k$ -vertex  $v$ ,  $k \geq 6$ , has non-negative initial charge. It receives charge from big faces (Rule 2) and sends charge to the hanging faces  $\gamma_1, \dots, \gamma_r$  (Rule 6). For each hanging face  $\gamma_i$ , the adjacent big face  $\beta_i$  is doubly-incident to  $v$ . The faces  $\beta_i$  and  $\beta_j$  are different for different  $\gamma_i$  and  $\gamma_j$  (see Claims 3.6 and 3.7). Therefore, the new charge of  $v$  is at least  $r \cdot (2 \cdot 1 - 2) = 0$ .

#### 3.3.2 1-faces

Let  $\gamma$  be a 1-face. It is adjacent to a big face  $\beta$  and it is hanging on a vertex  $v$ . The face  $\beta$  is adjacent to at most one 1-face (Claim 3.7). Therefore, it sends at least 2 units of charge to the face  $\gamma$  (Rule 1).

If the vertex  $v$  is a 2-vertex, we get the trivial graph, which is not a counterexample. If the vertex  $v$  is a 3-vertex, the third edge incident with  $v$  is a bridge in  $G$ , which is not allowed. Therefore,  $\deg(v) \geq 4$  and the vertex  $v$  sends 2 units of charge to the face  $\gamma$  (Rules 4 – 6). The new charge of  $\gamma$  is at least  $-4 + 2 + 2 = 0$ .

**3.3.3 2-faces**

Let  $\alpha$  be a 2-face. Its initial charge is  $-2$ . Let  $v_1$  and  $v_2$  be the vertices incident with  $\alpha$ . The face  $\alpha$  is adjacent to at most two faces, which must be big (see Claim 3.5). Consider the number of black vertices bad for  $\alpha$ . Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 8 for illustration.

1. Let both  $v_1$  and  $v_2$  be black and bad. Then the graph  $G$  consists of a single cycle on two vertices, which is not a counterexample.
2. Let  $v_1$  be a black 2-vertex. Then  $\alpha$  is adjacent to a single big face  $\beta$ . The big face  $\beta$  is not adjacent to any other small face (see Claim 3.14). Therefore,  $\beta$  sends 3 units of charge to  $\alpha$  (Rule 1). The face  $\alpha$  sends 2 units of charge to  $v_1$  and at most 1 unit of charge to  $v_2$  (Rule 3). On the other hand,  $\alpha$  is hanging on  $v_2$ , therefore, it receives 2 units of charge from  $v_2$  (Rules 4 – 6). Note that  $v_2$  cannot be a 3-vertex, otherwise there would be a bridge in  $G$ . The new charge of  $\alpha$  is at least  $-2 - 2 - 1 + 2 + 3 = 0$ .
3. Let none of  $v_1$  and  $v_2$  be black and bad. Consider the number of faces adjacent to  $\alpha$ . If  $\alpha$  is adjacent to a single big face  $\beta$ , then  $\beta$  is not adjacent to any other small face. Therefore,  $\beta$  sends 3 units of charge to  $\alpha$ . Moreover, in this case none of  $v_1$  and  $v_2$  can be neither black nor blue. The new charge of  $\alpha$  is  $-2 + 3 = 1$ .

If  $\alpha$  is adjacent to two big faces  $\beta_1$  and  $\beta_2$ , the face  $\alpha$  sends at most  $2 \cdot 1$  units of charge to  $v_1$  and  $v_2$ . The big face  $\beta_i, i = 1, 2$ , is not adjacent to any other small face of size at most 2 (see Claims 3.3, 3.4, and 3.6). Therefore, by Rule 1, the face  $\beta_i, i = 1, 2$ , sends at least 2 units of charge to  $\alpha$ . The new charge of  $\alpha$  is at least  $-2 - 2 \cdot 1 + 2 \cdot 2 = 0$ .

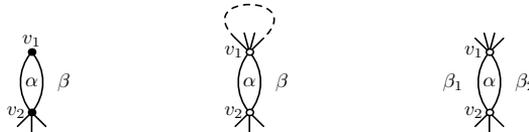


Figure 8: Different possible neighbourhoods of a 2-face  $\alpha$ .

**3.3.4 3-faces**

Let  $\alpha$  be a 3-face. Its initial charge is 0. Let  $v_1, v_2, v_3$  be the vertices incident with  $\alpha$ . (They do not have to be pairwise different.) Consider the number of black vertices bad for  $\alpha$ . Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 9 for illustration.

1. Let all the three vertices  $v_1, v_2, v_3$  be black and bad. Then the graph  $G$  consists of a single cycle on three vertices, which is not a counterexample.
2. Let  $v_2$  and  $v_3$  be black and bad. Then all the three edges of  $\alpha$  are incident with the same big face  $\beta$  and  $\alpha$  is hanging on  $v_1$ . Then by Claim 3.15 the face  $\beta$  sends 3 units of charge to  $\alpha$  (Rule 1). The face  $\alpha$  then sends  $2 \cdot 2$  units of charge to the 2-vertices  $v_2, v_3$  and at most 1 unit of charge to the vertex  $v_1$  (Rule 3). If the vertex  $v_1$  is a 3-vertex, the third edge incident with it is a bridge in  $G$ . Therefore

$\deg(v_1) \geq 4$  and  $v_1$  sends 2 units of charge to  $\alpha$  (Rules 4 – 6). The new charge of  $\alpha$  is  $3 - 2 \cdot 2 - 1 + 2 = 0$ .

- Let  $v_3$  be black and bad. Let  $\beta_1$  be the big face incident with the edge  $v_1v_2$  and  $\beta_2$  be the big face incident with the edges  $v_2v_3$  and  $v_3v_1$ .

The face  $\beta_2$  sends 3 units of charge to  $\alpha$ ,  $\beta_1$  sends at least 1 unit of charge to  $\alpha$ . The face  $\alpha$  then sends 2 units of charge to  $v_3$  and at most 1 unit of charge to  $v_1$  and  $v_2$ . The new charge of  $\alpha$  is at least  $3 + 1 - 2 - 2 \cdot 1 = 0$ .

- Let none of the vertices  $v_1, v_2, v_3$  be black and bad. Consider the number of faces adjacent to  $\alpha$ .

If there are three different faces adjacent to  $\alpha$  (they must be big, see Claim 3.5) then the face  $\alpha$  receives at least 1 unit of charge from each of them and sends at most 1 unit of charge to each incident vertex. Hence, the new charge of  $\alpha$  is at least 0.

If there is a big face  $\beta$  sharing at least two edges with  $\alpha$ , these edges are not face-adjacent in  $\beta$ . Therefore,  $\beta$  is not adjacent to any other small face, thus, it sends 3 units of charge to  $\alpha$ . The new charge of  $\alpha$  is non-negative again.

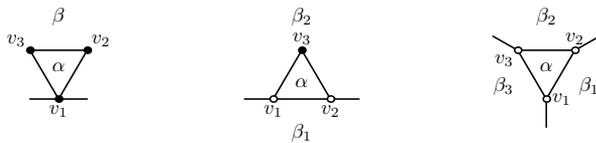


Figure 9: Different possible neighbourhoods of a 3-face  $\alpha$ .

### 3.3.5 4-faces

Let  $\alpha$  be a 4-face. Its initial charge is 2. Let  $v_1, v_2, v_3, v_4$  be the vertices incident with  $\alpha$ . (They do not have to be pairwise different.) Consider the number of black vertices bad for  $\alpha$ . Note that by Claim 3.13 each such vertex is a 2-vertex. See Figure 10 for illustration.

- Let all the four vertices  $v_1, v_2, v_3, v_4$  be black and bad. Then the graph  $G$  consists of a single cycle on four vertices, which is not a counterexample.
- Let  $v_1, v_2$ , and  $v_3$  be black and bad. Then all the four edges of  $\alpha$  are incident with the same big face  $\beta$  and  $\alpha$  is hanging on  $v_4$ . Hence,  $\beta$  sends 3 units of charge to  $\alpha$  (Rule 1). The face  $\alpha$  then sends  $3 \cdot 2$  units of charge to the 2-vertices  $v_1, v_2, v_3$  and at most 1 unit of charge to  $v_4$  (Rule 3). If the vertex  $v_4$  is a 3-vertex, the third edge incident with it is a bridge in  $G$ . Therefore,  $\deg(v_4) \geq 4$  and  $v_4$  sends 2 units of charge to  $\alpha$  (Rules 4 – 6). The new charge of  $\alpha$  is  $2 + 3 - 3 \cdot 2 - 1 + 2 = 0$ .
- Let  $v_1$  and  $v_3$  be black and bad. Let  $\beta_1$  be the big face incident with  $v_1$ , let  $\beta_2$  be the big face incident with  $v_3$ .

If  $\beta_1 \neq \beta_2$ , both  $\beta_1$  and  $\beta_2$  send 3 units of charge to  $\alpha$ . The face  $\alpha$  then sends 2 units of charge to the vertices  $v_1$  and  $v_3$  and at most 1 unit of charge to the vertices  $v_2$  and  $v_4$ . The new charge of  $\alpha$  is at least  $2 + 2 \cdot 3 - 2 \cdot 2 - 2 \cdot 1 = 2$ . If  $\beta_1 = \beta_2$ , then  $v_2$  and  $v_4$  are not blue, and the new charge of  $\alpha$  is at least  $2 + 3 - 2 \cdot 2 = 1$ .

4. Let  $v_1$  and  $v_2$  be black and bad. Let  $\beta_1$  be the big face incident with the edge  $v_3v_4$  and  $\beta_2$  be the big face incident with the vertices  $v_1$  and  $v_2$ . If  $\beta_1 \neq \beta_2$ ,  $\beta_2$  sends 3 units of charge to  $\alpha$  and  $\beta_1$  sends at least 1 unit of charge to  $\alpha$ . The face  $\alpha$  then sends 2 units of charge both to  $v_1$  and  $v_2$  and at most 1 unit of charge to  $v_3$  and  $v_4$ . The new charge of  $\alpha$  is at least  $2 + 3 + 1 - 2 \cdot 2 - 2 \cdot 1 = 0$ . If  $\beta_1 = \beta_2$ , then  $v_3$  and  $v_4$  are not blue, and the new charge of  $\alpha$  is at least  $2 + 3 - 2 \cdot 2 = 1$ .
5. Let  $v_1$  be black and bad. The big face  $\beta_1$  incident with  $v_1$  sends 3 units of charge to  $\alpha$ . The face  $\alpha$  then sends 2 units of charge to  $v_1$  and at most 1 unit of charge to  $v_2, v_3$ , and  $v_4$ . The new charge of  $\alpha$  is at least  $2 + 3 - 2 - 3 \cdot 1 = 0$ .
6. Let no black and bad vertex be incident with  $\alpha$ . Then the big faces adjacent to  $\alpha$  send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge), and  $\alpha$  sends at most 1 unit of charge to each incident vertex. The new charge of  $\alpha$  is at least  $2 + 2 - 4 = 0$ .

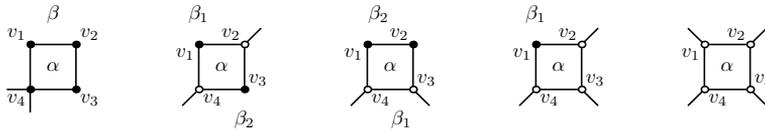


Figure 10: Different possible neighbourhoods of a 4-face  $\alpha$ .

### 3.3.6 5-faces

Let  $\alpha$  be a 5-face. Its initial charge is 4. Let  $v_1, v_2, v_3, v_4, v_5$  be the vertices incident with  $\alpha$ . (They do not have to be pairwise different.) Consider the number of black vertices bad for  $\alpha$ . Note that by Claim 3.11 each such vertex is either a 2-vertex or a 4-vertex with a hanging 1-face. See Figure 11 for illustration.

1. Let all the five vertices  $v_1, v_2, v_3, v_4, v_5$  be black and bad. Then the graph  $G$  contains only 5 vertices, hence, it is not a counterexample.
2. Let  $v_1, v_2, v_3$ , and  $v_4$  be black and bad. From Claim 3.6 it follows that none of them is incident with a 1-face. Then all the five edges of  $\alpha$  are incident with the same big face  $\beta$  and  $\alpha$  is hanging on  $v_5$ . Hence,  $\beta$  sends 3 units of charge to  $\alpha$  (Rule 1). The face  $\alpha$  then sends  $4 \cdot 2$  units of charge to the 2-vertices  $v_1, v_2, v_3$ , and  $v_4$  (Rule 3) and at most 1 unit of charge to the vertex  $v_5$  (Rule 3). If the vertex  $v_5$  is a 3-vertex, the third edge incident with it is a bridge in  $G$ . Therefore  $\deg(v_5) \geq 4$  and  $v_5$  sends 2 units of charge to  $\alpha$  (Rules 4 – 6). The new charge of  $\alpha$  is  $4 + 3 - 4 \cdot 2 - 1 + 2 = 0$ .
3. Let  $v_1, v_2$ , and  $v_3$  be black and bad. Let  $\beta_1$  be the big face incident with the vertices  $v_1, v_2$ , and  $v_3$  and  $\beta_2$  be the big face incident with the edge  $v_4v_5$ . By Claim 3.12  $v_1, v_2$ , and  $v_3$  are 2-vertices. If  $\beta_1 \neq \beta_2$ ,  $\beta_1$  sends 3 units of charge to  $\alpha$  and  $\beta_2$  sends at least 1 unit of charge to  $\alpha$ . The face  $\alpha$  then sends 2 units of charge to  $v_1, v_2$ , and  $v_3$  and at most 1 unit of charge to  $v_4$  and  $v_5$ . The new charge of  $\alpha$  is at least  $4 + 3 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$ . If  $\beta_1 = \beta_2$ , then  $v_4$  and  $v_5$  are not blue, and the new charge of  $\alpha$  is at least  $4 + 3 - 3 \cdot 2 = 1$ .

4. Let  $v_1, v_2$ , and  $v_4$  be black and bad. Let  $\beta_1$  be the big face incident with  $v_1$  and  $v_2$ , let  $\beta_2$  be the big face incident with  $v_4$ . By Claim 3.12  $v_1$  and  $v_2$  are 2-vertices, thus  $\beta_1$  sends 3 units of charge to  $\alpha$ .  
 If  $\beta_1 \neq \beta_2$ , then  $\beta_2$  sends at least 1 unit of charge to  $\alpha$ . The face  $\alpha$  then sends 2 units of charge to the vertices  $v_1, v_2$ , and  $v_4$  and at most 1 unit of charge to the vertices  $v_3$  and  $v_5$ . The new charge of  $\alpha$  is at least  $4 + 3 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$ . If  $\beta_1 = \beta_2$ , then  $v_3$  and  $v_5$  are not blue, and the new charge of  $\alpha$  is  $4 + 3 - 3 \cdot 2 = 1$ .
5. Let  $v_1$  and  $v_2$  be black and bad. Let  $\beta_1$  be the big face incident with the vertices  $v_1$  and  $v_2$ . The face  $\beta_1$  sends 3 units of charge to  $\alpha$ . The face  $\alpha$  then sends 2 units of charge both to  $v_1$  and  $v_2$  and at most 1 unit of charge to  $v_3, v_4$ , and  $v_5$ . The new charge of  $\alpha$  is at least  $4 + 3 - 2 \cdot 2 - 3 \cdot 1 = 0$ .
6. Let  $v_1$  and  $v_3$  be black and bad. Let  $\beta_1$  be the big face incident with  $v_1, \beta_2$  be the big face incident with  $v_3$ , and  $\beta_3$  be the big face incident with the edge  $v_4v_5$ . If  $\beta_1, \beta_2$ , and  $\beta_3$  are three different faces, the new charge of  $\alpha$  is at least  $4 + 3 - 2 \cdot 2 - 3 \cdot 1 = 0$ . If two of them coincide, then at least one of the vertices  $v_2, v_4$ , and  $v_5$  is not blue and the new charge of  $\alpha$  is at least  $4 + 2 - 2 \cdot 2 - 2 \cdot 1 = 0$ . If  $\beta_1 = \beta_2 = \beta_3$ , then  $v_2, v_4$ , and  $v_5$  are not blue and the new charge of  $\alpha$  is at least  $4 + 1 - 2 \cdot 2 = 1$ .
7. Let  $v_1$  be black and bad. Then the big faces adjacent to  $\alpha$  send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge). The new charge of  $\alpha$  is at least  $4 + 2 - 2 - 4 \cdot 1 = 0$ .
8. Let no black and bad vertex be incident with  $\alpha$ . Then the big faces adjacent to  $\alpha$  send together at least 1 unit of charge. The new charge of  $\alpha$  is at least  $4 + 1 - 5 = 0$ .

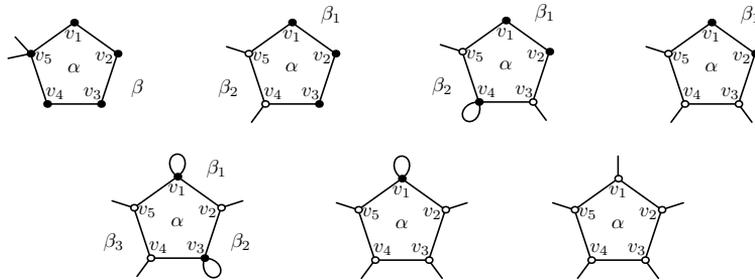


Figure 11: Different possible neighbourhoods of a 5-face  $\alpha$ .

### 3.3.7 Small faces of size at least 6

Let  $\alpha$  be a  $d$ -face,  $6 \leq d \leq 44$ . Its initial charge is  $2d - 6$ . Let  $v_1, \dots, v_d$  be the vertices incident with  $\alpha$ . (They do not have to be pairwise different.) Consider the black vertices incident with  $\alpha$ . Let  $v_i$  be a black 4-vertex. It cannot be good for  $\alpha$ , since no two small faces are adjacent (see Claim 3.5). Therefore, each black 4-vertex is bad for  $\alpha$ . By Claim 3.10 at most  $d - 2$  vertices incident with  $\alpha$  are bad. Let  $k \leq d - 2$  be the number of black vertices incident with  $\alpha$ . We can divide the facial walk of  $\alpha$  into  $d - k \geq 2$  parts, each beginning and ending in a blue or white vertex, each incident with  $\alpha$  and a big face  $\beta_i$ ,

$i \in \{1, \dots, d - k\}$ . Each of these big faces sends at least 1 unit of charge to  $\alpha$ . (If  $\beta_i = \beta_j$  for some  $1 \leq i < j \leq d - k$ , the face  $\beta_i$  cannot be adjacent to another small face but  $\alpha$ , therefore, it sends 3 units to  $\alpha$ , which is even more than what two different big faces would send.)

The face  $\alpha$  then sends 2 units of charge to each of the  $k$  incident black vertices, and at most 1 unit of charge to each of the other incident vertices. Together, the new charge of  $\alpha$  is at least

$$2d - 6 + (d - k) \cdot 1 - k \cdot 2 - (d - k) \cdot 1 = 2(d - k) - 6.$$

If  $d - k \geq 3$ , the new charge of  $\alpha$  is non-negative.

Let  $d - k = 2$ . It means there are only two vertices which are not black. Since  $d \geq 6$ , at least one big face  $\beta$  shares at least 2 black vertices with  $\alpha$ , say  $v_1$  and  $v_2$ . By Claim 3.10 at least one from  $v_1$  and  $v_2$  is a 2-vertex, hence, by Claim 3.14 the face  $\beta$  sends 3 units of charge to  $\alpha$ . The new charge of  $\alpha$  is therefore at least

$$2d - 6 + 3 + 1 - (d - 2) \cdot 2 - 2 \cdot 1 = 0.$$

### 3.3.8 Big faces

Let  $\beta$  be a  $d$ -face,  $d \geq 45$ . Its initial charge is  $2d - 6$ . It sends 3 units of charge to the small faces it is adjacent to (Rule 1). It sends 2 units of charge to all bad black vertices; 1 unit of charge to all other vertices. Let  $k$  be the number of black vertices bad for  $\beta$ . By Claim 3.10,  $k \leq \frac{4}{5} \cdot d$ . The new charge of  $\beta$  is therefore at least

$$2d - 6 - 3 - k \cdot 2 - (d - k) \cdot 1 = d - k - 9 \geq d - \frac{4d}{5} - 9 = \frac{d}{5} - 9 = \frac{d - 45}{5} \geq 0.$$

The new charge of all elements of the graph is non-negative, but the sum of all the charges is  $-12$ , which is a contradiction. This contradiction implies that the minimum counterexample does not exist.

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