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# A New Closed-Form Information Metric for Shape Analysis

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**Abstract.** Recently, a unifying framework was introduced for shape matching that uses mixture-models and the Fisher-Rao metric to couple both the shape representation and deformation. A fundamental drawback of the Fisher-Rao metric is that it is NOT available in closed-form for the mixture models making shape comparisons computationally very expensive. Here, we propose a new Riemannian metric based on generalized  $\phi$ -entropy measures. In sharp contrast to the Fisher-Rao metric, our new metric is available in closed-form.

## 1 Introduction

Shape analysis is a key ingredient to many medical imaging applications that seek to study the intimate relationship between the form and function of medical and biological structures. In recent work [1], it was shown that shape representation and deformation can be unified within a probabilistic framework – using a mixture of densities to represent landmark shapes and then directly using the Fisher information matrix to establish an intrinsic, Riemannian metric and subsequently a shape distance measure.

To address many of the computational inefficiencies that arise when using the standard Fisher-Rao information metric, the current work introduces a new Riemannian metric based on the generalized notion of a  $\phi$ -entropy functional.

## 2 Differential Metrics for Parametric Distributions

The parametric, GMM representation for 2-D shapes is given by

$$p(\mathbf{x}|\theta) = \frac{1}{2\pi\sigma^2 K} \sum_{a=1}^K \exp\left\{-\frac{\|\mathbf{x} - \phi_a\|^2}{2\sigma^2}\right\} \quad (1)$$

where  $\theta$  is our set of landmarks,  $\phi_a = [\theta^{(2a-1)}, \theta^{(2a)}]^T$ ,  $\mathbf{x} = [x^{(1)}, x^{(2)}]^T \in \mathbb{R}^2$  and equal weight priors are assigned to all components, i.e.  $\frac{1}{K}$ . Basically, a shape with  $K$  landmarks is represented as a  $K$ -component GMM where the landmarks are the means of each component.

Burbea and Rao demonstrated that the notion of distances between parametric models can be extended to a large class of generalized metrics [2] using the  $\phi$ -entropy functional

$$H_\phi(p) = - \int_{\chi} \phi(p) dx \quad (2)$$

where  $\chi$  is the measurable space (for our purposes  $\mathbb{R}^2$ ), and  $\phi$  is a  $C^2$ -convex function defined on  $\mathbb{R}_+ \equiv [0, \infty)$ . The metric on the parameter space is obtained by finding the Hessian of (2) along a direction in its tangent space. This directly leads to the following differential metric satisfying Riemannian metric properties

$$ds_\phi^2(\theta) = -\Delta_\theta H_\phi(p) = \sum_{i,j=1}^n g_{i,j}^\phi d\theta^i d\theta^j, \quad (3)$$

where

$$g_{i,j}^{\phi} = \int_{\mathcal{X}} \phi''(p) \left( \frac{\partial p}{\partial \theta^i} \right) \left( \frac{\partial p}{\partial \theta^j} \right) d\mathbf{x}. \quad (4)$$

The metric tensor in (4) is called the  $\phi$ -entropy matrix. We can define distances between probability densities by finding a geodesic between their parameter values as determined by (3).

The new metric can be obtained by selecting a  $\phi$  such that  $\phi''$  becomes a constant in (4). In [3], Havrda and Charvát introduced the notion of a  $\alpha$ -order entropy using the convex function

$$\phi(p) = (\alpha - 1)^{-1} (p^{\alpha} - p), \alpha \neq 1. \quad (5)$$

We set  $\alpha = 2$  which results in  $\frac{1}{2}\phi'' = 1$ . The one-half scaling factor does not impact the metric properties. The new metric is defined as

$$g_{i,j}^{\alpha} = \int_{\mathcal{X}} \left( \frac{\partial p}{\partial \theta^i} \right) \left( \frac{\partial p}{\partial \theta^j} \right) d\mathbf{x} \quad (6)$$

and we refer to it as the  $\alpha$ -order entropy metric tensor.

The new metric can be used to find the distance between two shapes represented as  $p(\mathbf{x}|\theta_1)$  and  $p(\mathbf{x}|\theta_2)$ . The geodesic can be obtained by solving

$$g_{ki}^{\alpha} \ddot{\theta}^i + \Gamma_{k,i,j} \dot{\theta}^i \dot{\theta}^j = 0. \quad (7)$$

Using this new metric, we now have closed-form solutions to  $g_{i,j}^{\alpha}$  and  $\frac{\partial g_{k,j}^{\alpha}}{\partial \theta^i}$ .

### 3 Experimental Results and Analysis

For applications in medical imaging, we have evaluated both metrics and compared to the standard thin-plate spline (TPS) bending energy on real data consisting of nine corpora callosa with 63-landmarks each. (Please see the conference paper for test results and analysis.)

The new  $\phi$ -entropy metric avoids an  $O(N^2)$  computational hit on the metric tensor computation and gains another  $O(N^2)$  savings when computing the derivative of the metric tensor.

### 4 Conclusion

In this paper, we introduced a new Riemannian metric for landmark-based shape matching that combines both shape representation and deformation. This improved on the previous work of [1] where the Fisher-Rao metric was used to establish an intrinsic metric between landmark shapes. The new  $\alpha$ -order entropy metric enables us to obtain closed-form solutions to the metric tensor and its derivatives. Our new approach also illustrated the possibilities of using information metrics besides Fisher-Rao and their benefits.

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