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Abstract. Recently, a unifying framework was introduced for shape matching that uses mixture-models and the Fisher-Rao metric to couple both the shape representation and deformation. A fundamental drawback of the Fisher-Rao metric is that it is NOT available in closed-form for the mixture models making shape comparisons computationally very expensive. Here, we propose a new Riemannian metric based on generalized ϕ- entropy measures. In sharp contrast to the Fisher-Rao metric, our new metric is available in closed-form.

1 Introduction

Shape analysis is a key ingredient to many medical imaging applications that seek to study the intimate relationship between the form and function of medical and biological structures. In recent work [1], it was shown that shape representation and deformation can be unified within a probabilistic framework – using a mixture of densities to represent landmark shapes and then directly using the Fisher information matrix to establish an intrinsic, Riemannian metric and subsequently a shape distance measure.

To address many of the computational inefficiencies that arise when using the standard Fisher-Rao information metric, the current work introduces a new Riemannian metric based on the generalized notion of a ϕ-entropy functional.

2 Differential Metrics for Parametric Distributions

The parametric, GMM representation for 2-D shapes is given by

$$p(\mathbf{x}|\theta) = \frac{1}{2\pi\sigma^2 K}\sum_{a=1}^{K}\exp\{-\frac{\|\mathbf{x}-\phi_a\|^2}{2\sigma^2}\} \tag{1}$$

where θ is our set of landmarks, $\phi_a = [\theta^{(2a-1)}, \theta^{(2a)}]^T$, $\mathbf{x} = [x^{(1)}, x^{(2)}]^T \in \mathbb{R}^2$ and equal weight priors are assigned to all components, i.e $\frac{1}{K}$. Basically, a shape with K landmarks is represented as a K-component GMM where the landmarks are the means of each component.

Burbea and Rao demonstrated that the notion of distances between parametric models can be extended to a large class of generalized metrics [2] using the *ϕ-entropy functional*

$$H_\phi(p) = -\int_\chi \phi(p)d\mathbf{x} \tag{2}$$

where χ is the measurable space (for our purposes $\mathbb{R}^2$), and ϕ is a C^2-convex function defined on $\mathbb{R}_+ \equiv [0,\infty)$. The metric on the parameter space is obtained by finding the Hessian of (2) along a direction in its tangent space. This directly leads to the following differential metric satisfying Riemannian metric properties

$$ds^2_\phi(\theta) = -\Delta_\theta H_\phi(p) = \sum_{i,j=1}^{n} g^\phi_{i,j}d\theta^i d\theta^j \ , \tag{3}$$

where

$$g^{\phi}_{i,j} = \int_{\chi} \phi''(p)(\frac{\partial p}{\partial \theta^i})(\frac{\partial p}{\partial \theta^j})d\mathbf{x} \; . \tag{4}$$

The metric tensor in (4) is called the ϕ-*entropy matrix*. We can define distances between probability densities by finding a geodesic between their parameter values as determined by (3).

The new metric can be obtained by selecting a ϕ such that ϕ'' becomes a constant in (4). In [3], Havrda and Charvát introduced the notion of a α-order entropy using the convex function

$$\phi(p) = (\alpha - 1)^{-1}(p^{\alpha} - p), \alpha \neq 1 \; . \tag{5}$$

We set $\alpha = 2$ which results in $\frac{1}{2}\phi'' = 1$. The one-half scaling factor does not impact the metric properties. The new metric is defined as

$$g^{\alpha}_{i,j} = \int_{\chi} (\frac{\partial p}{\partial \theta^i})(\frac{\partial p}{\partial \theta^j})d\mathbf{x} \tag{6}$$

and we refer to it as the α-*order entropy metric tensor*.

The new metric can be used to find the distance between two shapes represented as $p(\mathbf{x}|\theta_1)$ and $p(\mathbf{x}|\theta_2)$. The geodesic can be obtained by solving

$$g^{\alpha}_{ki}\ddot{\theta}^i + \Gamma_{k,ij}\dot{\theta}^i\dot{\theta}^j = 0. \tag{7}$$

Using this new metric, we now have closed-form solutions to $g^{\alpha}_{i,j}$ and $\frac{\partial g^{\alpha}_{kj}}{\partial \theta^i}$.

3 Experimental Results and Analysis

For applications in medical imaging, we have evaluated both metrics and compared to the standard thin-plate spline (TPS) bending energy on real data consisting of nine corpora callosa with 63-landmarks each. (Please see the conference paper for test results and analysis.)

The new ϕ-entropy metric avoids an $O(N^2)$ computational hit on the metric tensor computation and gains another $O(N^2)$ savings when computing the derivative of the metric tensor.

4 Conclusion

In this paper, we introduced a new Riemannian metric for landmark-based shape matching that combines both shape representation and deformation. This improved on the previous work of [1] where the Fisher-Rao metric was used to establish an intrinsic metric between landmark shapes. The new α-order entropy metric enables us to obtain closed-form solutions to the metric tensor and its derivatives. Our new approach also illustrated the possibilities of using information metrics besides Fisher-Rao and their benefits.

References

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