Economic receding horizon control without terminal constraints

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Abstract: We consider a receding horizon control scheme without terminal constraints in which the stage cost is defined by economic criteria, i.e., not necessarily linked to a stabilization or tracking problem. We analyze the performance of the resulting receding horizon controller with a particular focus on the case of optimal steady states for the corresponding averaged infinite horizon problem. Using a turnpike property and suitable controllability properties we prove near optimal performance of the controller and convergence of the closed loop solution to a neighborhood of the optimal steady state. Several examples illustrate our findings numerically and show how to verify the imposed assumptions.

1 Introduction

In this paper we investigate the performance of receding horizon control schemes with general stage costs. In receding horizon control — often also called model predictive control (MPC) — a feedback law is synthesized from the first elements of finite horizon optimal control sequences which are iteratively computed along the closed loop solution. This procedure has by now become a standard method for the optimization based stabilization and tracking control of linear and nonlinear systems. In stabilization problems, the stage cost of the underlying finite horizon optimal control problem typically penalizes the distance to a desired equilibrium or time varying reference solution. While there is an ample literature on the analysis of stabilizing receding horizon schemes — see, e.g., the survey paper [10] or the monographs [8, 12] and the extensive lists of references therein — results for schemes employing stage costs not related to stabilization and tracking are much more scarce. Due to the fact that in these schemes the stage cost ℓ usually reflects an economic criterion rather than a distance to a reference, they are often called economic MPC or economic receding horizon control.

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The receding horizon approach to economic problems is on the one hand appealing because it naturally yields a control function in feedback form. However, a probably even more important advantage is its ability to solve infinite horizon optimal control problems numerically with much lower computational effort than classical approaches like, e.g., dynamic programming. This is because the finite horizon problems have to be solved only locally in space and are thus much less affected by the curse of dimensionality, i.e., by the growth of computational complexity with increasing space dimension. Indeed, as, e.g., the examples in [8] show, using state-of-the-art optimization algorithms the method is nowadays able to handle even discretized PDE models with satisfactory accuracy.

When using receding horizon control in order to reduce the computational burden attached to infinite horizon optimal control problems, the question whether the resulting solution approximates the infinite horizon optimal solution becomes important. Indeed, while research in stabilizing MPC is often focused on issues like stability and feasibility, approximate optimality is the natural property to look at when the main objective is the optimization of a given cost criterion. While stability-like properties like the convergence to optimal steady states are of interest in economic problems, too, they are in general not a meaningful criterion on their own but only an additional feature once near optimal performance can be ensured. For economic receding horizon control schemes, these issues have recently been investigated in [1, 6, 2]. In these papers, conditions are derived under which the receding horizon closed loop shows optimal performance in terms of the averaged infinite horizon problem with the same stage cost. Moreover, conditions for the convergence of the closed loop solution towards an optimal equilibrium or a periodic solution are given. The central idea of the particular receding horizon approach in these references is as follows: first, one determines an optimal equilibrium or periodic orbit for the infinite horizon averaged problem and then this solution is used as a terminal constraint for the finite horizon optimal control problem to be solved in each step of the receding horizon scheme.

In contrast to these references, in this paper we do not impose any terminal constraints. Thus, roughly speaking, we investigate whether a receding horizon control scheme is able to find an optimal operating point or orbit "by itself", i.e., without having to compute this point or orbit a priori and providing it as additional information to the algorithm. This approach reduces the amount of preparatory computations which are needed in order to set up the scheme, simplifies the optimal control problem to be solved in each step and will often lead to a larger operating region of the resulting controller, because in the absence of terminal constraints we do not need to worry about the feasibility of these constraints for the finite horizon optimal control problem.

The price we pay for these simplifications is on the one hand a more involved analysis using stronger assumptions on the underlying finite horizon problems. To this end, we provide sufficient conditions based on a turnpike property and certain controllability assumptions which will be rigorously verified for a number of examples in this paper. On the other hand, our approach only yields approximate optimal performance instead of exact optimal performance as in the case of optimal equilibrium or periodic terminal constraints. However, we will prove that the performance converges to the optimal one as the receding optimization horizon grows and by numerical simulations we illustrate that the gap to optimality actually decreases rapidly for increasing optimization horizon. Moreover, the conditions we impose allow to prove approximate optimality of the receding horizon closed loop not only in an infinite horizon averaged sense but also in a finite horizon averaged sense during the transient phase, i.e., on the time interval until a neighborhood of the optimal steady state is reached. To the best of our knowledge results on approximately optimal transient behavior have not been obtained before in the economic MPC literature. While our general results are formulated in an abstract setting which covers various types of optimal solutions including certain types of periodic orbits, for the derivation of checkable sufficient conditions we focus on the particular case of optimal equilibria.

The paper is organized as follows. After formulating the problem and premilinary results in Section 2 we discuss three motivating examples in Section 3. These examples on the one hand illustrate the very good performance of the receding horizon scheme without stabilizing terminal constraints and on the other hand help to identify reasonable conditions to be imposed in the subsequent sections. General results on value convergence are given in Section 4, in which we first present a proposition which derives performance bounds from the existence of certain trajectories and then present a theorem which gives sufficient conditions on the finite horizon optimal trajectories and value functions under which such trajectories can be constructed. The conditions of this theorem are further investigated in Sections 5 and 6. Here we derive checkable sufficient conditions based on a turnpike property and suitable controllability conditions which can be rigorously checked in all our motivating examples. While Sections 5 and 6 mainly focus on the case of optimal steady states, in Section 7 we present an example which shows that our results also apply to certain classes of optimal periodic orbits. In Section 8 we investigate the limiting behavior of the receding horizon closed loop. Particularly, we give conditions for the convergence of the receding horizon closed loop solution towards (a neighborhood of) the infinite horizon optimal solution and show that the estimates derived in this section also imply approximate optimality during the transient phase, cf. Remark 8.7. Finally, Section 9 concludes the paper.

2 Problem formulation and preliminaries

We consider discrete time control systems with state $x \in X$ and control values $u \in U$, where X and U are normed spaces with norms denoted by $\|\cdot\|$. The control system under consideration is given by

$$x(k+1) = f(x(k), u(k)) \tag{2.1}$$

with $f : X \times U \to X$. For a given control sequence $u = (u(0), \ldots, u(K-1)) \in U^K$ or $u = (u(0), u(1), \ldots) \in U^\infty$, by $x_u(k, x)$ we denote the solution of (2.1) with initial value $x = x_u(0, x) \in X$.

For given admissible sets of states $\mathbb{X} \subseteq X$ and control values $\mathbb{U} \subseteq U$ and an initial value $x \in \mathbb{X}$ we call the control sequences $u \in \mathbb{U}^K$ satisfying

$$x_u(k, x) \in \mathbb{X} \quad \text{for all } k = 0, \ldots, K$$

admissible. The set of all admissible control sequences is denoted by $\mathbb{U}^K(x)$. Similarly, we define the set $\mathbb{U}^\infty(x)$ of admissible control sequences of infinite length. Since the emphasis of the analysis in this paper is on optimality rather than on feasibility, for simplicity of exposition we assume $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$, i.e., that for each initial value $x \in \mathbb{X}$ we can find a trajectory staying inside $\mathbb{X}$ for all future times. This condition may be relaxed if desired, using, e.g., the techniques from [8, Sections 8.2–8.3] or [11].

Given a feedback map $\mu : X \to U$, we denote the solutions of the closed loop system

$$x(k+1) = f(x(k), \mu(x(k)))$$

by $x_\mu(k)$ or by $x_\mu(k,x)$ if we want to emphasize the dependence on the initial value $x = x_\mu(0)$. We say that a feedback law μ is admissible if it renders the admissible set $\mathbb{X}$ (forward) invariant, i.e., if $f(x,\mu(x)) \in \mathbb{X}$ holds for all $x \in \mathbb{X}$. Note that $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$ immediately implies that such a feedback law exists.

Our goal is now to find an admissible feedback controller which yields trajectories with guaranteed bounds on the average cost, preferably as small as possible. To this end, for a given running cost $\ell : X \times U \to \mathbb{R}$ we define the averaged functionals and optimal value functions

$$\begin{array}{rclcrcl} J_N(x,u) & := & \displaystyle\frac{1}{N}\sum_{k=0}^{N-1} \ell(x_u(k,x),u(k)), & & V_N(x) & := & \displaystyle\inf_{u\in\mathbb{U}^N(x)} J_N(x,u), \\ J_\infty(x,u) & := & \displaystyle\limsup_{N\to\infty} J_N(x,u) & \text{and} & V_\infty(x) & := & \displaystyle\inf_{u\in\mathbb{U}^\infty(x)} J_\infty(x,u). \end{array}$$

Here we assume that ℓ is bounded from below on $\mathbb{X}$, i.e., that $\ell_{\min} := \inf_{x\in\mathbb{X},u\in\mathbb{U}} \ell(x,u)$ is finite. This assumption immediately yields $J_N(x,u) \ge \ell_{\min}$ and $J_\infty(x,u) \ge \ell_{\min}$ for all admissible control sequences. In order to simplify the exposition in what follows, we assume that (not necessarily unique) optimal control sequences for J_N exist which we denote by $u^*_{N,x}$. Formally, we assume that for each $x \in \mathbb{X}$ and each $N \in \mathbb{N}$ there exists $u^*_{N,x} \in \mathbb{U}^N(x)$ satisfying

$$V_N(x) = J_N(x, u^*_{N,x}).$$

Similarly to the open loop functionals, we can define the average cost of the closed loop solution for any feedback law μ by

$$\begin{aligned} J_K(x,\mu) &= \frac{1}{K}\sum_{k=0}^{K-1} \ell(x_\mu(k,x),\mu(x_\mu(k,x))) \\ J_\infty(x,\mu) &= \limsup_{K\to\infty} J_K(x,\mu). \end{aligned}$$

In order to find a feedback μ we will apply a receding horizon control scheme, also known as model predictive control (MPC). This method consists of solving the open loop optimization problem of minimizing $J_N(x,u)$ with initial value $x = x_\mu(k)$ at each sampling instant k for some given optimization horizon $N \in \mathbb{N}$ and then defining the feedback value $\mu(x) = \mu_N(x)$ to be the first element of the corresponding optimal control sequence, i.e.,

$$\mu_N(x) = u^*_{N,x}(0).$$

Since nowadays efficient algorithms for the necessary online minimization of $J_N(x,u)$ are available (see, e.g., [8, Chapter 10]), this method is computationally feasible for large classes of systems.

Our goal in this paper is to derive upper bounds for the functionals $J_K(x, \mu_N)$ and $J_\infty(x, \mu_N)$ depending on the optimization horizon N. While the upper bounds we deduce are in general not necessarily optimal, we are able to identify certain situations in which they actually are. Particularly, in the presence of optimal equilibria we can formulate checkable sufficient conditions for this property which are linked to the classical turnpike property and certain controllability assumptions, cf. the discussion at the end of Section 6. Still, our general setting does not necessarily need the existence of an optimal equilbrium, cf. e.g., Examples 5.7 and 7.1.

We end this section by introducing some basic notation and preliminary results. For subsets $\mathbb{Y} \subset X$ we denote the distance of a point $x \in X$ to $\mathbb{Y}$ by $|x|_{\mathbb{Y}} := \inf_{y \in \mathbb{Y}} \|x - y\|$. The open ball with radius $\delta > 0$ around a set $\mathbb{Y} \subset X$ will be denoted by $\mathcal{B}_\delta(\mathbb{Y}) := \{x \in X \mid |x|_{\mathbb{Y}} < \delta\}$. For sets consisting of one element $\mathbb{Y} = \{y\}$ we also write $\mathcal{B}_\delta(y)$ instead of $\mathcal{B}_\delta(\mathbb{Y})$. With $\mathcal{K}_\infty$ we denote the set of continuous functions $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which are strictly increasing and unbounded with $\alpha(0) = 0$. With $\mathcal{L}_\mathbb{N}$ we denote the set of functions $\delta : \mathbb{N} \to \mathbb{R}_0^+$ which are (not necessarily strictly) decreasing with $\lim_{k \to \infty} \delta(k) = 0$.

In our analysis we will make extensive use of the dynamic programming principle, cf. [3]. The form of this principle which applies here states that for the optimal control sequence $u^*_{N,x}$ for the problem with finite horizon N and each $K \in \{1, \dots, N-1\}$ the equality

$$V_N(x) = \frac{1}{N} \sum_{k=0}^{K-1} \ell(x_{u^*_{N,x}}(k,x), u^*_{N,x}(k)) + \frac{N-K}{N} V_{N-K}(x_{u^*_{N,x}}(K,x)) \tag{2.2}$$

holds. As a consequence, for $\mu_N(x) = u^*_{N,x}(0)$ we get

$$V_N(x) = \frac{1}{N}\ell(x, \mu_N(x)) + \frac{N-1}{N} V_{N-1}(f(x, \mu_N(x))).$$

This implies the equation

$$\ell(x, \mu_N(x)) = N V_N(x) - (N-1) V_{N-1}(f(x, \mu_N(x))). \tag{2.3}$$

3 Motivating examples

In order to illustrate how receding horizon control without terminal constraints performs for problems with economic cost, we look at three motivating examples. All simulations in this section and in the remainder of the paper were carried out with the MATLAB routine `nmpc.m` (cf. [8, Appendix A] and `www.nmpc-book.com`) which uses the `fmincon` optimization routine.

Example 3.1 (see also [7]) Consider the control system

$$x(k+1) = 2x(k) + u(k)$$

with $X = U = \mathbb{R}$ and $\mathbb{U} = [-2, 2]$. The running cost ℓ is chosen such that the control effort is penalized quadratically, i.e., $\ell(x, u) = u^2$ and we consider the admissible sets $\mathbb{X} = [-a, a]$

with $a = 0.5$ and $a = 1$. Hence, the optimal control problem consists of keeping the system inside the admissible set $\mathbb{X}$ with minimal average control effort, cf. also [7].

For this problem, it is easily seen that an optimal way of keeping the solutions inside $\mathbb{X}$ in the infinite horizon averaged sense is to steer the system to the equilibrium $x^e = 0$ in a finite number of steps k' and set $u(k) = u^e = 0$ for $k \geq k'$ which leads to $J_\infty(x, u) = 0$. Since $\ell(x, u) \geq 0$ for all x and u, this is the optimal value of J_∞, i.e., $V_\infty(x) = 0$ for all $x \in \mathbb{X}$.

As shown in [7], this example does not satisfy the usual conditions imposed on receding horizon control schemes in the literature. Nevertheless, the receding horizon feedback μ_N produces approximately optimal closed loop solutions. Figure 3.1 shows the infinite horizon averaged value $J_\infty(x, \mu_N)$ for the receding horizon strategy thus obtained for different optimization horizons N and the two admissible sets $\mathbb{X} = [-1, 1]$ (solid) and $\mathbb{X} = [-0.5, 0.5]$ (dashed). The values are plotted on a logarithmic scale and indicate that $J_\infty(x, \mu_N) \to 0$ exponentially fast as $N \to \infty$.

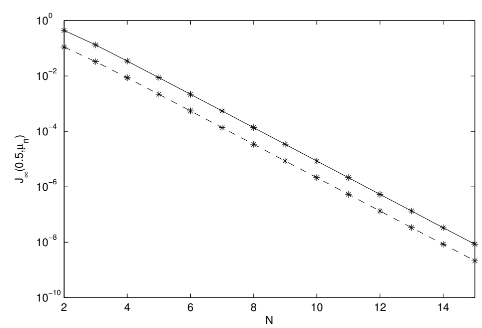

Figure 3.1: $J_\infty(x, \mu_N)$ for $N = 2, \ldots, 15$ and $x = 0.5$, $\mathbb{X} = [1, 1]$ (solid) and $\mathbb{X} = [-0.5, 0.5]$ (dashed)

We observe: for increasing optimization horizon N the closed loop infinite horizon averaged values $J_\infty(x, \mu_N)$ improve and approach the optimum $V_\infty(x) = 0$ as $N \to \infty$. On the other hand, for the larger admissible set $\mathbb{X} = [-1, 1]$ the values are larger — despite the fact that the infinite horizon optimal value does not depend on the choice of $\mathbb{X}$.

Figure 3.2 shows the corresponding closed loop trajectories for $\mathbb{X} = [-0.5, 0.5]$ with optimization horizon $N = 5$ (solid) and $N = 10$ (dashed).

It is interesting to compare the closed loop trajectories with the optimal open loop trajectories in each step of the scheme, as illustrated in Figure 3.3 for $\mathbb{X} = [-1, 1]$ and $N = 5$. While the closed loop trajectory approaches a neighborhood of $x^* = 0$, the optimal open loop trajectories tend towards the upper boundary $x = 1$ of the admissible set $\mathbb{X} = [-1, 1]$.

□

Example 3.2 The second example is a simple one dimensional economic growth model introduced in [4], see also [9]. While the problem was considered in discounted form in these references, here we consider a version with averaged optimality criterion. Furthermore, we

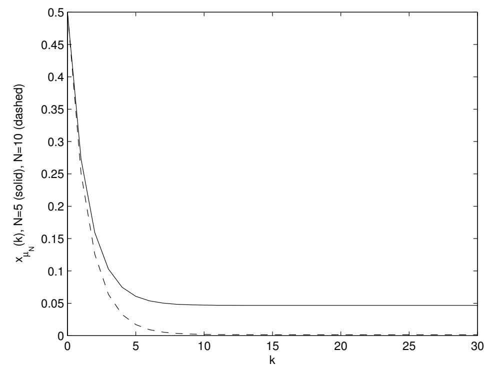

Figure 3.2: $x_{\mu_N}(k,x)$ for $N = 5$ (solid) and $N = 10$ (dashed), both for $x = 0.5$ and $\mathbb{X} = [-0.5, 0.5]$

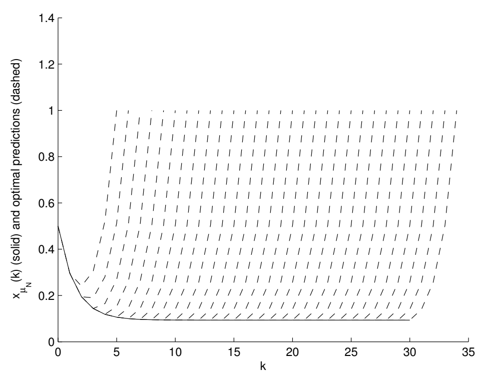

Figure 3.3: Optimal predictions $x_{u^*_{N,x}}(k, x_{\mu_N}(k))$ (dashed) within the receding horizon optimization for $N = 5$, $x = 0.5$ and $\mathbb{X} = [-1, 1]$

change the sign of the stage cost in order to convert the original maximization problem into a minimization problem such that it fits our setting. The dynamics of the system is

$$x(k+1) = u(k)$$

and the stage cost is

$$\ell(x,u) = -\ln(Ax^\alpha - u).$$

Here we use the parameters $A = 5$ and $\alpha = 0.34$ and impose the state constraints $\mathbb{X} = [0, 10]$ and the control constraints $\mathbb{U} = [0.1, 5]$. Observing that each $x \in \mathbb{U} = [0.1, 5]$ is an equilibrium for the (unique) control value $u = x$, the optimal equilibrium value of the stage cost is

$$\ell^e := \min_{x \in \mathbb{U}} \ell(x,x),$$

which (with the help of Maple) evaluates to $\ell^e = -\ln(3) - \ln(11) + 50\ln(2)/33 + 50\ln(5)/33 - 17\ln(17)/33 \approx -1.467276$ and is attained at $x^e = 17^{17/33}10^{16/33}17/100 \approx 2.234421$. Figure

3.4(left) shows the MPC closed loop trajectory (solid) and the optimal predicted open loop trajectory (dashed) for each sampling instant for $N = 5$ and initial value $x_0 = 5$.

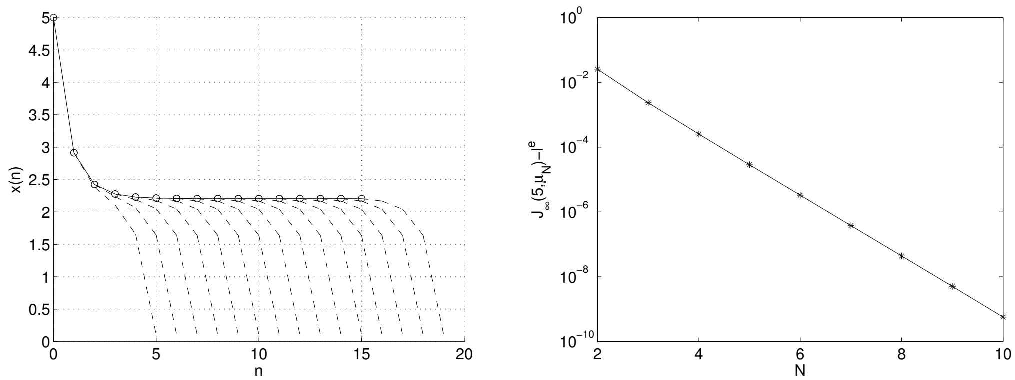

Figure 3.4: Closed loop trajectory $x_{\mu_N}(k, x_0)$ (solid) and optimal predictions $x_u(k, x_{\mu_N}(k))$ (dashed) for Example 3.2 for $N = 5$ and $x_0 = 5$ (left) and $J_\infty(5, \mu_N) - \ell^e$ for $N = 2, \dots, 10$ (right)

Here, the open loop trajectories first approach and then stay near a value of $x \approx x^e$ for a while before they eventually tend to 0. The resulting MPC closed loop trajectory converges to $x \approx x^e$. Since the trajectory converges to a point near x^e, one may expect that the infinite horizon optimal cost is also close to the optimal equilibrium cost ℓ^e. Figure 3.4(right), in which the difference $J_\infty(5, \mu_N) - \ell^e$ is plotted for different optimization horizons N on a logarithmic scale confirms that this is indeed the case; more precisely, the difference decreases exponentially for growing N. The figure also shows that even for rather small optimization horizons the difference between the MPC average closed loop value $J_\infty(5, \mu_N)$ and the optimal equilibrium value ℓ^e is very small. $\square$

Example 3.3 Our final example in this section is an unmatched setpoint example taken from [1], where it was considered with terminal constraints. The dynamics is again one-dimensional with dynamics and cost function

$$x(k+1) = 0.5x(k) + 15u(t) - 7.5, \qquad \ell(x, u) = \begin{cases} 50u^2 + \delta x^2, & x > 0 \\ 0.25x^2 + 50u^2 + 3.34xu, & x < 0. \end{cases}$$

where $\delta > 0$ is a small value in order to make the cost function strictly convex; in our computations we chose $\delta = 0.01$. The goal of the x-part of cost function is to keep the state at $x^* = 0$, however, since the control needed for staying at this equilibrium is $u^* = 0.5$, the cost in this equilibrium is $\ell(x^*, u^*) = 12.5$ and thus this is not the optimal equilibrium in terms of the cost function. In fact, the optimal equilibrium is given by $x^e = -2145/536 \approx -4.002$ with control $u^e = 393/1072$ and $\ell^e = \ell(x^e, u^e) = 124857/21440 \approx 5.824$, which can be seen as a compromise between the distance to the desired state $x^* = 0$ and the necessary control effort, measured in terms of the stage cost.

Figure 3.5(left) shows the MPC closed loop trajectory (solid) and the optimal predicted open loop trajectory (dashed) for each sampling instant for $N = 5$ and initial value $x_0 = 5$. For the computations we imposed the state and control constraints $\mathbb{X} = \mathbb{U} = [-10, 10]$.

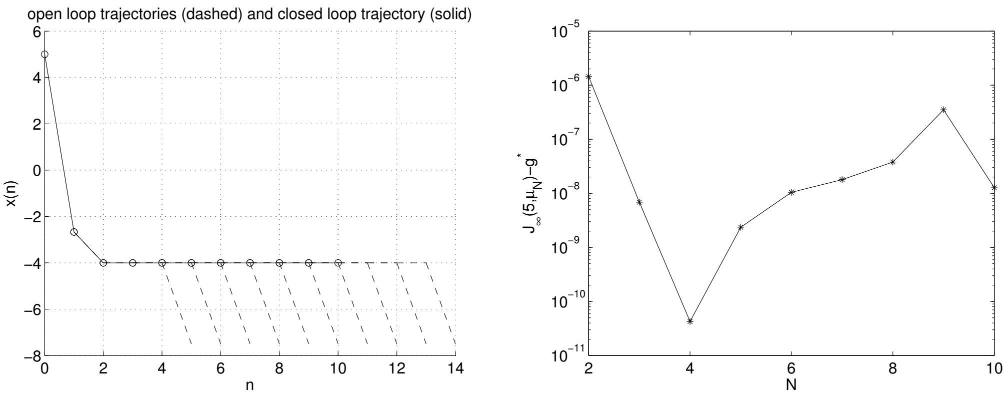

Figure 3.5: Closed loop trajectory $x_{\mu_N}(k, x_0)$ (solid) and optimal predictions $x_u(k, x_{\mu_N}(k))$ (dashed) for Example 3.2 for $N = 5$ and $x_0 = 5$ (left) and $J_\infty(5, \mu_N) - \ell^e$ for $N = 2, \ldots, 10$ (right)

The behavior is similar to Figure 3.4(left) in Example 3.2: the open loop trajectories first approach and then stay near a value of $x \approx x^e$ for a while before they eventually tend to a value near -8. The resulting MPC closed loop trajectory converges to $x \approx x^e$. Again, one may thus expect that the infinite horizon optimal cost is also close to the optimal equilibrium cost ℓ^e. Figure 3.5(right) shows the difference $J_\infty(5, \mu_N) - \ell^e$ is plotted for different optimization horizons N on a logarithmic scale. Again, the figure confirms that the averaged value of the MPC closed loop is close to the optimal equilibrium value, although in this example numerical errors in the optimization routine become visible for $N \geq 5$ which prevent the convergence to be as nice as in Figure 3.1 or 3.4(right). Still, the values are very close to the optimal equlibrium value ℓ^e. □

4 Value convergence

Our goal in this section is to investigate the dependence of $J_\infty(x, \mu_N)$ on N. The following Proposition 4.1 gives an upper bound for this value. Its proof uses the classical receding horizon proof technique to prolong a suitable control sequence of length N in order to obtain a sequence of length $N+1$ for which the difference between J_{N+1} and V_N can be estimated. However, since we have seen in Figure 3.3 that the optimal trajectories for the finite horizon problem end up at the boundary of the admissible set, in the setting considered in this paper it is in general not efficient to construct a suitable prolonged control sequence by adding an additional element at the end. Instead, we use control sequences in which an additional element is inserted at an arbitrary place into the control sequence. In the following theorem we assume that a suitably extended control sequence with an additional element inserted at time $k_{N,x}$ has already been constructed and formulate conditions on this sequence under which we can derive estimates on $J_\infty(x, \mu_N)$. Sufficient conditions under which such a sequence can be constructed will then be introduced in the subsequent Theorem 4.2. In order to facilitate this construction, in Proposition 4.1 we do not assume optimality of the control sequence and corresponding trajectory without the additional element, but rather only approximate optimality with a suitable bound on the error term.

Proposition 4.1 Assume there are $N_0 > 0$ and $\delta_1, \delta_2 \in \mathcal{L}_{\mathbb{N}}$ such that for each $x \in \mathbb{X}$ and $N \geq N_0$ there exists a control sequence $u_{N,x} \in \mathbb{U}^{N+1}$ and $k_{N,x} \in \{0, \ldots, N\}$ satisfying the following conditions.

(i) The inequality

$$J_N'(x) := \frac{1}{N} \sum_{\substack{k=0 \\ k \neq k_{N,x}}}^{N} \ell(x_{u_{N,x}}(k,x), u_{N,x}(k)) \leq V_N(x) + \delta_1(N)/N$$

holds.

(ii) There exists $\ell_0 \in \mathbb{R}$ such that for all $x \in \mathbb{X}$ the inequality

$$\ell(x_{u_{N,x}}(k_{N,x}, x), u_{N,x}(k_{N,x})) \leq \ell_0 + \delta_2(N)$$

holds.

Then the inequalities

$$J_K(x, \mu_N) \leq \frac{N}{K} V_N(x) - \frac{N}{K} V_N(x_{\mu_N}(K)) + \ell_0 + \delta_1(N-1) + \delta_2(N-1) \tag{4.1}$$

and

$$J_\infty(x, \mu_N) \leq \ell_0 + \delta_1(N-1) + \delta_2(N-1) \tag{4.2}$$

hold for all $x \in \mathbb{X}$, all $N \geq N_0 + 1$ and all $K \in \mathbb{N}$.

Proof: Fix $x \in \mathbb{X}$ and $N \geq N_0 + 1$. Abbreviating $x(k) = x_{\mu_N}(k, x)$, from (2.3) for any $k \geq 0$ we get

$$\frac{1}{K} \ell(x(k), \mu_N(x(k))) = \frac{N}{K} V_N(x(k)) - \frac{N-1}{K} V_{N-1}(x(k+1)).$$

Summing up for $k = 0, \ldots, K-1$ then yields

$$\begin{aligned} J_K(x, \mu_N) &= \frac{1}{K} \sum_{k=0}^{K-1} \ell(x(k), \mu_N(x(k))) \\ &= \sum_{k=0}^{K-1} \left(\frac{N}{K} V_N(x(k)) - \frac{N-1}{K} V_{N-1}(x(k+1)) \right) \\ &= \frac{N}{K} V_N(x(0)) - \frac{N-1}{K} V_{N-1}(x(K)) \\ &\quad + \frac{1}{K} \sum_{k=1}^{K-1} \Big(N V_N(x(k)) - (N-1) V_{N-1}(x(k)) \Big). \end{aligned} \tag{4.3}$$

Now we investigate the terms in (4.3). Property (i) with $N-1$ in place of N and $x = x(k)$ implies

$$(N-1)V_{N-1}(x(k)) \quad \geq \quad (N-1)\widetilde{J}_{N-1}(x(k)) - \delta_1(N-1).$$

Furthermore, by optimality of V_N we get

$$V_N(x(k)) \leq J_N(x(k), u_{N-1,x(k)}).$$

Combining these inequalities, using the definition of J_N and J'_N and (ii), for the summands of (4.3) we get

$$\begin{aligned}
&NV_N(x(k)) - (N-1)V_{N-1}(x(k)) \\
&\quad \leq \; NJ_N(x(k), u_{N-1,x(k)}) - (N-1)J'_{N-1}(x(k)) + \delta_1(N-1) \\
&\quad = \; \ell(x_{u_{N-1,x(k)}}(k_{N-1,x(k)}, x(k_{N-1,x(k)})), u_{N-1,x(k)}(N-1)) + \delta_1(N-1) \\
&\quad \leq \; \ell_0 + \delta_2(N-1) + \delta_1(N-1).
\end{aligned} \tag{4.4}$$

Recalling that $x(0) = x$ and inserting (4.4) for $k = 1, \ldots, K-1$ into (4.3) yields

$$\begin{aligned}
J_K(x, \mu_N) \quad \leq \quad & \frac{N}{K}V_N(x) - \frac{N-1}{K}V_{N-1}(x(K)) \\
& + \frac{K-1}{K}\Big(\ell_0 + \delta_2(N-1) + \delta_1(N-1)\Big).
\end{aligned}$$

Using (4.4) for $k = K$ and dividing by K furthermore yields

$$-\frac{N-1}{K}V_{N-1}(x(K)) \leq -\frac{N}{K}V_N(x(K)) + \frac{1}{K}\Big(\ell_0 + \delta_2(N-1) + \delta_1(N-1)\Big).$$

Thus, we get

$$J_K(x, \mu_N) \leq \frac{N}{K}V_N(x) - \frac{N}{K}V_N(x(K)) + \ell_0 + \delta_2(N-1) + \delta_1(N-1),$$

i.e., (4.1). Inequality (4.2) follows from (4.1) by letting $K \to \infty$ since $V_N(x(K)) \geq \ell_{\min}$. □

In order to check whether the conditions of Proposition 4.1 are satisfied for a given example, we need to be able to construct the control sequence $u_{N,x}$ of length $N+1$ from an optimal control sequence with horizon N by inserting an additional element at time k_x. The following theorem gives conditions on the finite horizon optimal value functions and trajectories under which such a construction is possible. Its statement is constructive in the sense that $u_{N,x}$ meeting the assumptions of Proposition 4.1 is explicitly constructed in the proof and the conditions imposed in the proposition can be rigorously checked for all our motivating examples, as shown in the subsequent sections.

Theorem 4.2 Assume that there exists a set $\mathbb{Y} \subseteq \mathbb{X}$ and a value $\ell_0 \geq 0$ such that for each $x \in \mathbb{Y}$ there is a control value $u \in \mathbb{U}$ with $f(x, u) \in \mathbb{Y}$ and $\ell(x, u) \leq \ell_0$. Assume furthermore that there exist $\bar{\delta} > 0$ such that the following properties hold.

(a) There exists γ_f, $\gamma_\ell \in \mathcal{K}_\infty$ such that for all $\delta \in (0,\bar\delta]$ and all $x \in \mathcal{B}_\delta(\mathbb{Y})$ there is $u_x \in \mathbb{U}$ such that $f(x,u_x) \in \mathbb{X}$ and the inequalities

$$|f(x,u_x)|_{\mathbb{Y}} \le \gamma_f(\delta) \quad \text{and} \quad \ell(x,u_x) \le \ell_0 + \gamma_\ell(\delta)$$

hold.

(b) There exists $N_0 \in \mathbb{N}_0$ and $\gamma_V \in \mathcal{K}_\infty$ such that for all $\delta \in (0,\bar\delta]$, all $N \in \mathbb{N}$ with $N \ge N_0$ and all $x \in \mathcal{B}_\delta(\mathbb{Y})$ and $y \in \mathbb{Y}$ the inequality

$$|V_N(x) - V_N(y)| \le \gamma_V(\delta)/N$$

holds.

(c) There exists $\sigma \in \mathcal{L}_\mathbb{N}$ and $N_1 \in \mathbb{N}$ with $N_1 \ge N_0$ for $N_0 \in \mathbb{N}_0$ from (b), such that for each $x \in \mathbb{X}$ and each $N \ge N_1$ there exists an optimal trajectory $x_{u^*_{N,x}}(\cdot,x)$ satisfying $|x_{u^*_{N,x}}(k_x,x)|_{\mathbb{Y}} \le \sigma(N)$ for some $k_x \in \{0,\dots,N-N_0\}$.

Then there exists $N_2 \in \mathbb{N}$ such that the inequalities

$$J_K(x,\mu_N) \le \frac{N}{K}V_N(x) - \frac{N}{K}V_N(x_{\mu_N}(K)) + \ell_0 + \varepsilon(N-1) \tag{4.5}$$

and

$$J_\infty(x,\mu_N) \le \ell_0 + \varepsilon(N-1) \tag{4.6}$$

hold for all $x \in \mathbb{X}$, $K \in \mathbb{N}$, all $N \ge N_2 + 1$ and $\varepsilon \in \mathcal{L}_\mathbb{N}$ given by $\varepsilon(N) = \gamma_V(\sigma(N)) + \gamma_V(\gamma_f(\sigma(N))) + \gamma_\ell(\sigma(N))$. If, in addition,

$$V_N(y) \le V_N(x) \tag{4.7}$$

holds for all $N \in \mathbb{N}$, $y \in \mathbb{Y}$ and $x \in \mathcal{B}_{\bar\delta}(\mathbb{Y})$, then the assertion holds for $\varepsilon(N) = \gamma_V(\gamma_f(\sigma(N))) + \gamma_\ell(\sigma(N))$.

Proof: We first prove the general case and discuss the modifications for the case (4.7) at the end of the proof. We show that the assumptions of Proposition 4.1 hold for $\delta_1(N) = \gamma_V(\sigma(N)) + \gamma_V(\gamma_f(\sigma(N)))$ and $\delta_2(N) = \gamma_\ell(\sigma(N))$ and then use this theorem in order to conclude the assertion. Note that $\delta_1, \delta_2 \in \mathcal{L}_\mathbb{N}$ and thus also $\varepsilon \in \mathcal{L}_\mathbb{N}$.

To establish the assumptions of Proposition 4.1, we choose $N_2 \ge N_1$ such that $\sigma(N_2) \le \bar\delta$ and $\gamma_f(\sigma(N_2)) \le \bar\delta$ holds for σ from (c) and γ_f from (a). Now pick $N \ge N_2$, $x \in \mathbb{X}$ and the corresponding optimal control $u^*_{N,x} \in \mathbb{U}^N(x)$ from (c). Let k_x be the time index from (c), abbreviate $x' = x_{u^*_{N,x}}(k_x,x)$ and let $u_{x'}$ be the control value from (a) for $x = x'$. Let $x'' = f(x',u_{x'})$ and let $u^*_{N-k_x,x''}$ be an optimal control sequence for initial value $x = x''$ and horizon $N - k_x$. Using these values, we define the control sequence $u_{N,x} \in \mathbb{U}^{N+1}(x)$ by

$$u_{N,x}(k) := \begin{cases} u^*_{N,x}(k), & k = 0,\dots,k_x - 1 \\ u_{x'}, & k = k_x \\ u^*_{N-k_x,x''}(k - k_x - 1), & k = k_x + 1,\dots,N. \end{cases} \tag{4.8}$$

This implies

$$x_{u_{N,x}}(k,x) = x_{u^*_{N,x}}(k,x), \qquad \text{for } k = 0,\dots,k_x,$$
$$|x'|_{\mathbb{Y}} \le \sigma(N), \quad |x''|_{\mathbb{Y}} = |f(x',u_{x'})|_{\mathbb{Y}} \le \gamma_f(\sigma(N)) \tag{4.9}$$

and

$$\ell(x',u_{x'}) \le \ell_0 + \gamma_\ell(\sigma(N)). \tag{4.10}$$

Using the fact that (b) implies $V_N(y) = V_N(y')$ for all $y, y' \in \mathbb{Y}$, from (4.9) and (b) it follows that we can pick an arbitrary $y \in \mathbb{Y}$ in order to conclude the inequality

$$\begin{aligned} V_K(x'') &\le V_K(y) + \frac{\gamma_V(\gamma_f(\sigma(N)))}{K} \\ &\le V_K(x') + \frac{\gamma_V(\sigma(N)) + \gamma_V(\gamma_f(\sigma(N)))}{K} \;=\; V_K(x') + \frac{\delta_1(N)}{K} \end{aligned} \tag{4.11}$$

for any $K \in \mathbb{N}$ with $K \ge N_0$. By (c) we have that $K = N - k_x \ge N_0$. Now we distinguish two cases:

In case $N - k_x \ge 1$ we can use (4.11) with $K = N - k_x \ge N_0$ in order to obtain

$$\begin{aligned} \frac{1}{N-k_x}\sum_{k=k_x+1}^{N} \ell(x_{u_{N,x}}(k,x), u_{N,x}(k)) &= J_{N-k_x}(x'', u^*_{N-k_x,x''}) \;=\; V_{N-k_x}(x'') \\ &\le V_{N-k_x}(x') + \frac{\delta_1(N)}{N-k_x}. \end{aligned} \tag{4.12}$$

Setting $k_{x,N} = k_x$ in Assumption (i) of Proposition 4.1, in case $N - k_x \ge 1$ we then obtain

$$\begin{aligned} J'_N(x) &= \frac{1}{N}\sum_{\substack{k=0\\k\ne k_x}}^{N} \ell(x_{u_{N,x}}(k,x), u_{N,x}(k)) \\ &= \frac{1}{N}\sum_{k=0}^{k_x-1} \ell(x_{u_{N,x}}(k,\tilde u_x(k)) + \frac{1}{N}\sum_{k=k_x+1}^{N} \ell(x_{u_{N,x}}(k,u_{N,x}(k)) \\ &= V_N(x) - \frac{N-k_x}{N} V_{N-k_x}(x') + \frac{N-k_x}{N}\frac{1}{N-k_x}\sum_{k=k_x+1}^{N} \ell(x_{u_{N,x}}(k,u_{N,x}(k)) \\ &\le V_N(x) - \frac{N-k_x}{N} V_{N-k_x}(x') + \frac{N-k_x}{N}\left(V_{N-k_x}(x') + \frac{\delta_1(N)}{N-k_x}\right) \\ &= V_N(x) + \delta_1(N)/N, \end{aligned}$$

where we have used (2.2) in the third step and (4.12) in the fourth step. This shows Assumption (i) of Proposition 4.1 with $\delta_1(N) = \gamma_V(\sigma_N) + \gamma_V(\gamma_f(\sigma(N)))$.

In case $N - k_x = 0$ we obtain

$$J'_N(x) = \frac{1}{N}\sum_{\substack{k=0\\k\ne k_x}}^{N} \ell(x_{u_{N,x}}(k,x), u_{N,x}(k)) = \frac{1}{N}\sum_{k=0}^{N-1} \ell(x_{u_{N,x}}(k,\tilde u_x(k)) \;=\; V_N(x)$$

and thus Assumption (i) of Proposition 4.1 holds with arbitrary $\delta_1(N)$. Hence, in both cases Assumption (i) of Proposition 4.1 holds with $\delta_1(N) = \gamma_V(\sigma_N) + \gamma_V(\gamma_f(\sigma(N)))$.

Furthermore, from (4.10) we get

$$\ell(x_{u_{N,x}}(k_x, x), u_{N,x}(k_x)) = \ell(x', u_{x'}) \leq \ell_0 + \gamma_\ell(\sigma(N)),$$

i.e., Assumption (ii) of Proposition 4.1 with $\delta_2(N) = \gamma_\ell(\sigma(N))$. Thus, Proposition 4.1 applies and (4.5) and (4.6) follow with $\varepsilon(N) = \delta_1(N)+\delta_2(N) = \gamma_V(\sigma(N))+\gamma_V(\gamma_f(\sigma(N)))+\gamma_\ell(\sigma(N))$. This finishes the proof in the general case.

In case (4.7) holds, we can improve (4.11) to

$$V_K(x'') \leq V_K(y) + \frac{\gamma_V(\gamma_f(\sigma(N)))}{K} \leq V_K(x') + \frac{\gamma_V(\gamma_f(\sigma(N)))}{K}$$

which shows that we can choose $\delta_1(N) = \gamma_V(\gamma_f(\sigma(N))$ in this case. Continuing as above with this modified δ_1 shows the assertion with $\varepsilon(N) = \delta_1(N) + \delta_2(N) = \gamma_V(\gamma_f(\sigma(N))) + \gamma_\ell(\sigma(N))$. □

While Condition (a) from Theorem 4.2 is quite easy to check using continuity of f and ℓ, Conditions (b) and (c) are much more difficult to verify. In the next two sections we will thus discuss checkable sufficient conditions for Conditions (b) and (c). We start with Condition (c).

5 Optimal steady states and the turnpike property

Condition (c) demands that the optimal solution "passes by" near the set $\mathbb{Y}$. In this section we mainly investigate this property for the important special case where $\mathbb{Y} = \{x^e\}$ is an equilibrium. We will in particular derive a checkable sufficient condition based on the so called turnpike property and an asymptotic controllability condition. Some remarks on more general cases of $\mathbb{Y}$ are given at the end of the section. We start with the following definition which defines infinite horizon optimality of an optimal steady state or equilibrium.

Definition 5.1 A pair $(x^e, u^e) \in \mathbb{X}\times\mathbb{U}$ is called an *equilibrium* or *steady state* if $f(x^e, u^e) = x^e$ holds. For a given steady state and stage cost ℓ we say that the system is *optimally operated at steady state* if for each initial value $x \in \mathbb{X}$ and each admissible control sequence $u \in \mathbb{U}^\infty(x)$ the inequality

$$\liminf_{N\to\infty} J_N(x, u) \geq \ell(x^e, u^e)$$

holds. □

A checkable sufficient condition for this property is obtained by the following procedure taken from [2]. We define a modified cost

$$\tilde{\ell}(x, u) := \ell(x, u) + \lambda(x) - \lambda(u) \tag{5.1}$$

for a given function $\lambda : \mathbb{X} \to \mathbb{R}$. One easily verifies that the inequality $\min_{x\in\mathbb{X},u\in\mathbb{U}} \tilde{\ell}(x^e,u^e) \leq \tilde{\ell}(x^e,u^e) = \ell(x^e,u^e)$ holds. Now we assume that the identity

$$\min_{x\in\mathbb{X},u\in\mathbb{U}} \tilde{\ell}(x,u) = \ell(x^e,u^e) \tag{5.2}$$

holds, a property referred to as dissipativity in [2]. One easily checks that this property is satisfied for Example 3.1 for $\lambda \equiv 0$. A little computation (with the help of Maple) reveals that (5.2) holds for Example 3.2 with $\lambda(x) = \sigma x$ with $\sigma = 17^{16/33}10^{17/33}10/561 \approx 0.230553$. For Example 3.3, inequality (5.2) holds for $\lambda(x) = \sigma x$ with $\sigma = 41619/26800 \approx 1.552948$. Thus, for both examples we can choose λ as a linear function which is also the approach taken in [6] and [5, Section 4.4].

The following result is a modified version of [2, Theorem 2] to our setting without terminal constraints.

Theorem 5.2 Assume that there exists $\lambda : \mathbb{X} \to \mathbb{R}^n$ satisfying (5.2) for some steady state $(x^e,u^e) \in \mathbb{X}\times\mathbb{U}$ and $C := 2\sup_{x\in\mathbb{X}} |\lambda(x)| < \infty$. Then the system is optimally operated at the steady state (x^e,u^e).

Proof: We define

$$\widetilde{J}_N(x,u) := \frac{1}{N}\tilde{\ell}(x_u(k,x),u(k)).$$

Then one easily checks that for each $x \in \mathbb{X}$ and $u \in \mathbb{U}^N(x)$ the identity

$$\widetilde{J}_N(x,u) = J_N(x,u) + \frac{\lambda(x)}{N} - \frac{\lambda(x_u(N,x)}{N} \tag{5.3}$$

holds. Since u is admissible we get $x \in \mathbb{X}$ and $x_u(N,u) \in \mathbb{X}$ and we can conclude

$$\widetilde{J}_N(x,u) \leq J_N(x,u) + \frac{C}{N}. \tag{5.4}$$

On the other hand, due to (5.2) we immediately get $\widetilde{J}_N(x,u) \geq \ell(x^e,u^e)$. This yields

$$\liminf_{N\to\infty} J_N(x,u) \geq \liminf_{N\to\infty} \widetilde{J}_N(x,u) - \frac{C}{N} \geq \ell(x^e,u^e)$$

and thus shows the assertion. □

Since in all our examples a continuous λ satisfying (5.2) can be found and $\mathbb{X}$ is bounded, hence $|\lambda(x)|$ is bounded over $\mathbb{X}$, this theorem is applicable.

An immediate consequence of Definition 5.1 is the following corollary.

Corollary 5.3 Assume that the system is optimally operated at the steady state x^e, u^e and let the assumptions of Theorem 4.2 hold for $\mathbb{Y} = \{x^e\}$. Then there exists $N_2 \in \mathbb{N}$ such that the inequality

$$J_\infty(x,\mu_N) \leq V_\infty(x) + \varepsilon(N-1) \tag{5.5}$$

holds for all $x \in \mathbb{X}$ and all $N \geq N_2 + 1$ for $\varepsilon \in \mathcal{L}_{\mathbb{N}}$ specified in Theorem 4.2. In particular, this implies the identity

$$\lim_{N\to\infty} J_\infty(x,\mu_N) = V_\infty(x) \tag{5.6}$$

for all $x \in \mathbb{X}$.

Proof: From Definition 5.1 it follows that $V_\infty(x) \geq \ell(x^e, u^e)$ holds. On the other hand, the choice of $\mathbb{Y}$ immediately implies that the value ℓ_0 in Theorem 4.2 can be chosen as $\ell_0 = \ell(x^e, u^e)$. Now (5.5) follows directly from Theorem 4.2 and (5.6) follows since $\varepsilon \in \mathcal{L}_\mathbb{N}$ and the obvious inequality $J_\infty(x, \mu_N) \geq V_\infty(x)$. □

Now we proceed with the derivation of a sufficient condition for Theorem 4.2(c). In order to link the optimality at steady states to this condition we use the following stronger version of (5.2).

There exists $\gamma \in \mathcal{K}_\infty$ such that the inequality

$$\min_{u\in\mathbb{U}} \tilde{\ell}(x,u) \geq \ell(x^e, u^e) + \alpha_\ell(\|x - x^e\|) \tag{5.7}$$

holds for all $x \in \mathbb{X}$. Again with the help of Maple one checks that this condition is satisfied for the functions λ specified above for Example 3.2 and 3.3. It is not satisfied for Example 3.1 with $\lambda \equiv 0$, but one can check that it holds, e.g., for $\lambda(x) = x^2/2$. Figure 5.1, which shows $\min_{u\in\mathbb{U}} \tilde{\ell}(x,u)$ for the three examples, also clearly indicates that (5.7) is satisfied.

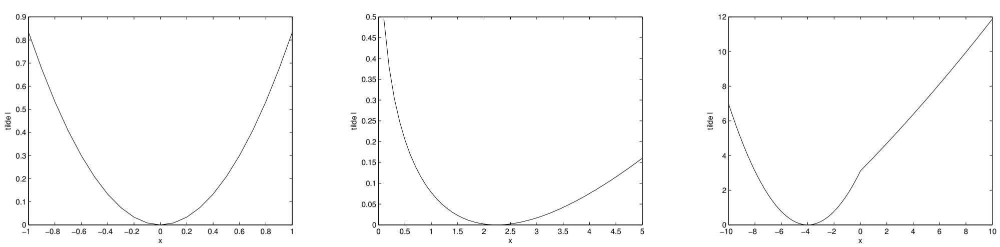

Figure 5.1: The functions $x \mapsto \min_{u\in\mathbb{U}} \tilde{\ell}(x,u)$ for Example 3.1, 3.2 and 3.3 (left to right)

The following theorem shows a consequence from (5.7) which is known as the turnpike property, cf. [5, Section 4.4]. Here we present it in a discrete time version and provide a quantitative estimate for the value Q_ε.

Theorem 5.4 Assume that there exists $\lambda : \mathbb{X} \to \mathbb{R}^n$ satisfying (5.7) for some $\alpha_\ell \in \mathcal{K}_\infty$ and a steady state $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ and $C := 2\sup_{x\in\mathbb{X}} |\lambda(x)| < \infty$. Then for each $x \in \mathbb{X}$, each $\delta > 0$, each control sequence $u \in \mathbb{U}^N(x)$ satisfying $J(x,u) \leq \ell(x^e, u^e) + \delta/N$ and each $\varepsilon > 0$ the value

$$Q_\varepsilon := \#\{k \in \{0, \ldots, N-1\} \mid \|x_u(k,x) - x^e\| \leq \varepsilon\}$$

satisfies the inequality $Q_\varepsilon \geq N - (\delta + C)/\alpha_\ell(\varepsilon)$.

Proof: Using (5.4) we obtain the inequality

$$\widetilde{J}_N(x,u) \leq J_N(x,u) + \frac{C}{N} \leq \ell(x^e, u^e) + \frac{\delta + C}{N}. \tag{5.8}$$

Now assume that $Q_\varepsilon < N - (\delta + C)/\alpha_\ell(\varepsilon)$. This means that there exists a set $\mathcal{N} \subseteq \{0, \ldots, N-1\}$ of $N - Q_\varepsilon > (\delta + C)/\alpha_\ell(\varepsilon)$ times instants such that $\|x_u(k,x) - x^e\| > \varepsilon$

holds for all $k \in \mathcal{N}$. This implies

$$\begin{aligned}
\widetilde{J}_N(x,u) &= \frac{1}{N}\sum_{k\in\mathcal{N}} \underbrace{\ell(x_u(k,x),u(k))}_{\geq \ell(x^e,u^e)+\alpha_\ell(\varepsilon)} + \frac{1}{N}\sum_{k\in\{0,\ldots,N-1\}\setminus\mathcal{N}} \underbrace{\ell(x_u(k,x),u(k))}_{\geq \ell(x^e,u^e)} \\
&\geq \frac{N-Q_\varepsilon}{N}(\ell(x^e,u^e)+\alpha_\ell(\varepsilon)) + \frac{Q_\varepsilon}{N}\ell(x^e+u^e) \\
&= \ell(x^e,u^e) + \frac{N-Q_\varepsilon}{N}\alpha_\ell(\varepsilon) \\
&> \ell(x^e,u^e) + \frac{(\delta+C)/\alpha_\ell(\varepsilon)}{N}\alpha_\ell(\varepsilon) \;=\; \ell(x^e,u^e)+\frac{\delta+C}{N}.
\end{aligned}$$

This contradicts (5.8) and thus proves the theorem. □

The last ingredient we need in order to conclude Condition (c) of Theorem 4.2 from the turnpike property is an asymptotic controllability property with respect to the stage cost ℓ. For its formulation we need the following subclass of $\mathcal{KL}$-functions.

Definition 5.5 We define the class $\mathcal{KLS}$ as the class of summable $\mathcal{KL}$ functions which sum up to a $\mathcal{K}$ function, i.e., as the class of functions $\beta \in \mathcal{KL}$ for which the infinite sum $\sum_{k=0}^\infty \beta(r,k)$ is finite for all $r \geq 0$ and for which $\gamma_\beta : \mathbb{R}_0^+ \to \mathbb{R}$ given by

$$\gamma_\beta(r) := \sum_{k=0}^\infty \beta(r,k)$$

satisfies $\gamma_\beta \in \mathcal{K}$. □

The asymptotic controllability property we need is now as follows: We assume that there exists $\beta \in \mathcal{KLS}$ such that for each $x \in \mathbb{X}$ and each $N \in \mathbb{N}$ there is a control function $u \in \mathbb{U}^N(x)$ such that the inequality

$$\ell(x_u(k,x),u(k)) \leq \ell(x^e,u^e) + \beta(\|x-x^e\|,k) \tag{5.9}$$

holds for all $k = 0,\ldots,N-1$.

This property again holds for all examples from Section 3. Indeed, in all examples it is possible to steer the system to x^e in one step using an appropriate control value u_x and the corresponding values of ℓ becomes the smaller the closer x is to x^e. Defining the control sequence $u = (u_x, u^e, u^e, \ldots)$ then yields (5.9) where β can be chosen as summable because we have $\ell(x_u(k,x),u(k)) = \ell(x^e,u^e)$ for all $k \geq 1$.

With this property we can now prove the main theorem of this section.

Theorem 5.6 Assume that there exists $\lambda : \mathbb{X} \to \mathbb{R}^n$ satisfying (5.7) for some $\alpha_\ell \in \mathcal{K}_\infty$ and a steady state $(x^e,u^e) \in \mathbb{X}\times\mathbb{U}$ and $C := 2\sup_{x\in\mathbb{X}}|\lambda(x)| < \infty$. Assume, moreover, that $\mathbb{X}$ is bounded and that (5.9) holds. Then Condition (c) of Theorem 4.2 holds for $\mathbb{Y} = \{x^e\}$.

Proof: The asymptotic controllability assumption immediately yields the inequality

$$J_N(x,u) \le \ell(x^e,u^e) + \frac{1}{N}(\gamma_\beta(\|x - x^e\|).$$

Hence, since $\mathbb{X}$ and thus $\|x - x^e\|$ is bounded for all $x \in \mathbb{X}$ we get the inequality

$$V_N(x) \le \ell(x^e,u^e) + \frac{\delta}{N}$$

with $\delta = \gamma_\beta(\max_{x\in\mathbb{X}}(\|x - x^e\|)$. Now we choose $N_1 = N_0$ and set $\sigma(N)$ arbitrary for $N \le N_1$ and

$$\sigma(N) := \alpha_\ell^{-1}\left(\frac{\delta + C}{N - N_0}\right)$$

otherwise, with $\alpha_\ell \in \mathcal{K}_\infty$ from (5.7). Clearly, this function lies in $\mathcal{L}_\mathbb{N}$ because as $N \to \infty$ the argument of α_ℓ^{-1} tends to 0 and thus α_ℓ^{-1} does so, too, since inverse functions of $\mathcal{K}_\infty$ functions are again $\mathcal{K}_\infty$ functions. This choice of σ implies

$$Q_{\sigma(N)} \ge N - \frac{\delta + C}{\alpha_\ell(\sigma(N))} = N_0.$$

Hence, there are at least N_0 time instants k for which $\|x_u(k,x) - x^e\| \le \sigma(N)$ holds and consequently at least one of these k must satisfy $k \in \{0,\dots,N-N_0\}$. Condition (c) thus holds if we choose k_x as this k. □

Since we have shown during this section that all examples from Section 3 satisfy the conditions of Theorem 5.6, we have thus proved that all examples satisfy Condition (c) of Theorem 4.2. The proof of Theorem 5.6 moreover reveals why the closed loop value is larger for larger $\mathbb{X}$. Since both δ and C depend on $\mathbb{X}$, both constants grow as $\mathbb{X}$ becomes larger. Consequently, $\sigma(N)$ in Condition (c) increases and thus $\varepsilon(N-1)$ in (4.6) increases, too. We like to remark, however, that the construction in the proof of Theorem 5.6 does not necessarily yield the smallest possible $\sigma(N)$. In fact, in our examples α_ℓ is a polynomial which implies that $\sigma(N)$ from the proof satisfies $\sigma(N) = O(1/N^\kappa)$ for some $\kappa > 0$. Numerical evaluation of the optimal trajectories, on the other hand, shows that in all our examples $\sigma(N)$ decays exponentially in N.

While we conjecture that many arguments in this section can be generalized from the case $\mathbb{Y} = \{x^e\}$ to arbitrary sets $\mathbb{Y}$, for the sake of brevity we restrict ourselves to a brief discussion of an extension of Corollary 5.3 by means of two examples. It is an easy exercise to show that this corollary remains true for general subsets $\mathbb{Y} \subseteq \mathbb{X}$ if the right hand side $\ell(x^e,u^e)$ in the inequality in Definition 5.1 is replaced by ℓ_0 from Proposition 4.2. The following two examples show the opportunities and limitations of this generalization.

Example 5.7 We consider the state and control value space with $X = U = \mathbb{R}$ and admissible sets consisting of only two elements $\mathbb{X} = \mathbb{U} = \{0,1\}$. We use the dynamics $x(k+1) = u(k)$ and the cost function

$$\ell(0,0) := 1, \;\; \ell(0,1) := 0, \;\; \ell(1,0) := 0 \quad \text{and} \quad \ell(1,1) = 1.$$

Clearly, for each $N \ge 1$ the optimal control sequence can be given in feedback form as $u(k) = 1$ if $x(k) = 0$ and $u(k) = 0$ if $x(k) = 1$, hence the open loop optimal trajectories jump periodically from 0 to 1 and back. The resulting MPC feedback law is $\mu_N(x) = 1$ if $x = 0$ and $\mu_N(x) = 0$ if $x = 1$ and the resulting closed loop value is $J_\infty(x, \mu_N) = 0$ for each $x \in \mathbb{X}$ which is also the optimal value $V_\infty(x)$ because $\ell(x, u) \geq 0$ for all $x \in \mathbb{X}$, $u \in \mathbb{U}$.

Due to the discrete nature of the state space Conditions (a) and (b) of Theorem 4.2 are trivially satisfied for $\overline{\delta} = 1/2$ and since the open loop optimal trajectories are periodic Condition (c) is satisfied for any nonempty set $\mathbb{Y} \subseteq \mathbb{X}$. However, for $\mathbb{Y} = \{0\}$ or $\mathbb{Y} = \{1\}$ any ℓ_0 meeting the assumptions must satisfy $\ell_0 \geq 1$ and thus cannot be used in order to conclude $\lim_{N\to\infty} J_\infty(x, \mu_N) = 0$. The only choice for which we can get the optimal value $\ell_0 = V_\infty(x) = 0$ is $\mathbb{Y} = \mathbb{X} = \{0, 1\}$. The reason for this behavior lies in the fact that the system is not optimally operated at any of the steady states 0 or 1. Rather, the optimal solutions need to jump periodically between the two steady states and the more flexible choice of $Y = \{0, 1\}$ allows to take this fact into account. □

Example 5.8 While in Example 5.7 it is possible to obtain a tight estimate for $\mathbb{Y} = \{0, 1\}$, by slightly modifying the example we can construct the situation in which the identity $\lim_{N\to\infty} J_\infty(x, \mu_N) = V_\infty(x)$ holds but cannot be concluded from Theorem 4.2 regardless of how $\mathbb{Y}$ is chosen. Changing ℓ in Example 5.7 to

$$\ell(0,0) := 1, \;\; \ell(0,1) := 0, \;\; \ell(1,0) := 1/2 \quad \text{and} \quad \ell(1,1) = 1$$

one easily sees that the optimal trajectories remain the same for all $N \geq 1$ and that the infinite horizon optimal value which coincides with the MPC closed loop value equals $V_\infty(x) = J_\infty(x, \mu_N) = 1/4$. However, regardless of how $\mathbb{Y}$ is chosen it is impossible to meet the conditions of Proposition 4.2 with $\ell_0 < 1/2$. □

Remark 5.9 These examples show that one can in general only conclude the tight upper bound $\lim_{N\to\infty} J_\infty(x, \mu_N) = V_\infty(x)$ from Proposition 4.2 if the system is optimally operated at steady state or if the stage cost is constant along the infinite horizon optimal trajectories in $\mathbb{Y}$. While these are admittedly special situations, they cover the important case of optimal equilibria as well as certain types of periodic behavior as illustrated by Example 7.1, below. Generalizations of our results to larger classes of optimal orbits will be investigated in future research. □

6 Controllability conditions

In the last section we have seen that the turnpike property together with an asymptotic controllability property can be used in order to conclude Condition (c) of Theorem 4.2. In this section we show a similar result for Condition (b) of Theorem 4.2. Again, we focus on the special case $\mathbb{Y} = \{x^e\}$. We present a sufficient condition for the required continuity property of V_N based on the turnpike property and the following local controllability condition:

Assumption 6.1 There exists $\delta_c > 0$, $d \in \mathbb{N}$ and $\gamma_x, \gamma_u, \gamma_c \in \mathcal{K}_\infty$ such that for each trajectory $x_{u_1}(k, x)$ with $u_1 \in \mathbb{U}^d(x)$ satisfying $x_u(k, x) \in \mathcal{B}_{\delta_c}(x^e)$ for all $k = 0, \ldots, d$ and all $x_1, x_2 \in \mathcal{B}_{\delta_c}(x^e)$ there exists $u_2 \in \mathbb{U}^d(x)$ satisfying

$$x_{u_2}(d, x_1) = x_2$$

and the estimates

$$\|x_{u_2}(k,x_1) - x_{u_1}(k,x)\| \le \gamma_x(\max\{\|x_1 - x\|, \|x_2 - x_{u_1}(d,x)\|\}),$$
$$\|u_2(k) - u_1(k)\| \le \gamma_u(\max\{\|x_1 - x\|, \|x_2 - x_{u_1}(d,x)\|\})$$

and

$$|\ell(x_{u_2}(k,x_1), u_2(k)) - \ell(x_{u_1}(k,x), u_1(k)| \le \gamma_c(\max\{\|x_1 - x\|, \|x_2 - x_{u_1}(d,x)\|\})$$

for all $k = 0, \dots, d-1$. □

This assumption is satisfied for all examples from Section 3 with $d = 1$. More precisely, for Example 3.1 we can choose $u_2(0) = x_2 - 2x_1$, for Example 3.2 we set $u_2(0) = x_2$ and for Example 3.3 we choose $u_2(0) = (x_2 - x_1/2 + 7.5)/15$. One checks that the distance of these control values to $u_1(0)$ is proportional to the distance of x_1 and x_2 to x and $x_{u_1}(1,x)$, respectively. This yields the existence of γ_x and γ_u and from this the existence of γ_c follows by continuity of ℓ in x and u. Moreover, one can verify that for sufficiently small $\delta_c > 0$ the value $u_1(0)$ must be in the interior of $\mathbb{U}$, thus $u_2(0)$ is also contained in $\mathbb{U}$ and is thus admissible.

More generally, we conjecture that for systems with $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ we can conclude Asumption 6.1 by linearization techniques if the linearization of f in (x^e, u^e) is controllable and the trajectory $x_{u_1}(k,x)$ and the values $u_1(k)$ lie in the interior of $\mathbb{X}$ and $\mathbb{U}$, respectively.

The following lemma shows an important consequence from Assumption 6.1 and the turnpike property. For its proof we need another assumption on the modified stage cost $\tilde{\ell}$, namely the existence of $\alpha_u \in \mathcal{K}_\infty$ such that the inequality

$$\tilde{\ell}(x,u) \le \ell(x^e, u^e) + \alpha_u(\|x - x^e\| + \|u - u^e\|) \tag{6.1}$$

holds. This property is again satisfied for all examples for the modified stage costs from the last section because these are Lipschitz in x and u and satisfy $\tilde{\ell}(x^e,u^e) = \ell(x^e,u^e)$.

Lemma 6.2 Let the asumptions of Theorem 5.4 and Assumption 6.1 as well as (6.1) hold. Then there exists $N_1 > 0$, a function $P : \mathbb{N} \to \mathbb{N}$ with $P(N) \ge N/2$ and $\eta : \mathbb{N} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\eta(N,r) \to 0$ if $N \to \infty$ and $r \to 0$ such that the open loop optimal trajectories with horizon $N \ge N_1$ starting in $x_1 \in \mathcal{B}_{\delta_c}(x_e)$ satisfy

$$\|x_{u^*_{N,x_1}}(k,x_1) - x^e\| \le \eta(N, \|x_1 - x^e\|)$$

for all $k = 0, \dots, P(N)$.

Proof: Using Assumption 6.1 with x_1 from the assumption, $x = x_2 = x^e$ and $u_1 \equiv u^e$ we get the estimate

$$J_d(x_1, u_2) \le \ell(x^e, u^e) + \gamma_c(\|x_1 - x^e\|)$$

and $x_{u_2}(d, u_2) = x^e$. Picking $N \ge d$ and extending the control sequence u_2 by setting $u_2(k) = u^e$ for $k = d, \dots, N$ we thus obtain

$$J_N(x_1, u^*_{N,x_1}) \le J_N(x_1, u_2) \le \ell(x^e, u^e) + \frac{d}{N}\gamma_c(\|x_1 - x^e\|).$$

Hence, we can apply Theorem 5.4 to $x = x_1$ and $u = u_2$ with $\delta = d\gamma_c(\|x_1 - x^e\|)$.

We pick $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $Q_\varepsilon \geq 2d$ holds in Theorem 5.4. We set $P(N)$ to be the largest k such that $\|x_{u^*_{N,x_1}}(k, x_1) - x^e\| \leq \varepsilon$ holds. With this choice, $P(N) \geq Q_\varepsilon \geq 2d$ holds. Now we use Assumption 6.1 with $x_1 = x = x^e$, $u_1 \equiv u^e$ and $x_2 = x_{u^*_{N,x_1}}(P(N), x_1)$ and denote the resulting control sequence by $\bar{u}_2$. This sequence satisfies

$$J_d(x^e, \bar{u}_2) \leq \ell(x^e, u^e) + \gamma_c(\varepsilon).$$

Now we pick the control sequence u_2 from the beginning of the proof and define a new control sequence $\bar{u}$ via

$$\bar{u}(k) = \begin{cases} u_2(k), & k = 0, \dots, d-1 \\ u^e, & k = d, \dots, P(N) - d - 1 \\ \bar{u}_2(k - P(N) + d), & k = P_N - d, \dots, P(N) - 1 \\ u^*_{N,x_1}(k), & k = P(N), \dots, N-1. \end{cases}$$

By construction of $\bar{u}$ the corresponding trajectory satisfies

$$x_{\bar{u}}(k, x_1) = \begin{cases} x_{u_2}(k, x_1), & k = 0, \dots, d \\ x^e, & k = d, \dots, P(N) - d \\ x_{\bar{u}_2}(k - P(N) + d, x^e), & k = P_N - d, \dots, P(N) \\ x_{u^*_{N,x_1}}(k, x_1), & k = P(N), \dots, N. \end{cases}$$

Using the fact that by the optimality principle the last piece of the trajectory is optimal for horizon $N - P(N)$, we obtain

$$J_N(\bar{u}, x_1) = \frac{P(N)}{N} J_{P(N)}(\bar{u}, x_1) + \frac{N - P(N)}{N} V_{N-P(N)}(x_{u^*_{N,x_1}}(P(N), x_1))$$

and

$$J_N(u^*_{N,x_1}, x_1) = \frac{P(N)}{N} J_{P(N)}(u^*_{N,x_1}, x_1) + \frac{N - P(N)}{N} V_{N-P(N)}(x_{u^*_{N,x_1}}(P(N), x_1)).$$

Subtracting the second from the first inequality and using the optimality of $J_N(u^*_{N,x_1}, x_1)$ implies

$$J_{P(N)}(u^*_{N,x_1}, x_1) \leq J_{P(N)}(\bar{u}, x_1).$$

Moreover, since $x_{\bar{u}}(P(N), x_1) = x_{u^*_{N,x_1}}(P(N), x_1)$, by (5.3) we get

$$J_N(u^*_{N,x_1}, x_1) - J_{P(N)}(\bar{u}, x_1) = \widetilde{J}_N(u^*_{N,x_1}, x_1) - \widetilde{J}_{P(N)}(\bar{u}, x_1)$$

and thus

$$\widetilde{J}_{P(N)}(u^*_{N,x_1}, x_1) \leq \widetilde{J}_{P(N)}(\bar{u}, x_1). \tag{6.2}$$

From the construction of $\bar{u}$ via Assumption 6.1 we now get the estimates

$$\|x_{\bar{u}}(k, x_1) - x^e\| \leq \gamma_x(\|x_1 - x^e\|) \quad \text{and} \quad \|\bar{u}(k) - u^e\| \leq \gamma_u(\|x_1 - x^e\|)$$

for $k = 0, \dots, d-1$ and

$$\|x_{\bar{u}}(k, x_1) - x^e\| \leq \gamma_x(\varepsilon) \quad \text{and} \quad \|\bar{u}(k) - u^e\| \leq \gamma_u(\varepsilon)$$

for $k = P(N)-d+1,\dots,P(N)$, while for $k = d,\dots,P(N)-d$ we get $x_{\bar u}(k,x_1) = x^e$ and $\bar u(k) = u^e$. Using (6.1), for the modified functional this implies

$$\widetilde J_{P(N)}(\bar u, x_1) \le \ell(x^e,u^e) + \frac{d}{P(N)}\alpha_u(\gamma_x(\|x_1-x^e\|)+\gamma_u(\|x_1-x^e\|)) + \frac{d}{P(N)}\alpha_u(\gamma_x(\varepsilon)+\gamma_u(\varepsilon)).$$

On the other hand, if we assume that $\|x_{u^*_{N,x_1}}(k,x_1)-x^e\| \ge \Delta$ for some $\Delta > 0$ and some $k \in \{0,\dots,P(N)-1\}$, then from (5.7) we get

$$\widetilde J_{P(N)}(x_1, u^*_{N,x_1}) \ge \ell(x^e,u^e) + \frac{\alpha_\ell(\Delta)}{P(N)}.$$

Hence, in case that

$$\Delta > \alpha_\ell^{-1}(d\alpha_u(\gamma_x(\|x_1-x^e\|)+\gamma_u(\|x_1-x^e\|)) + d\alpha_u(\gamma_x(\varepsilon)+\gamma_u(\varepsilon)))$$

we get the inequality

$$\widetilde J_{P(N)}(x_1, u^*_{N,x_1}) > \widetilde J_{P(N)}(\bar u, x_1)$$

which contradicts (6.2). Thus, we get

$$\Delta \le \alpha_\ell^{-1}(d\alpha_u(\gamma_x(\|x_1-x^e\|)+\gamma_u(\|x_1-x^e\|)) + d\alpha_u(\gamma_x(\varepsilon)+\gamma_u(\varepsilon))).$$

The assertion now follows by choosing $\varepsilon = \alpha_\ell^{-1}(2(\delta+C)/N)$ which implies $P(N) \ge Q_\varepsilon \ge N/2$ as well as $\varepsilon \to 0$ as $N \to \infty$. Setting $N_1 = 4d$ then ensures $P(N) \ge 2d$ for $N \ge N_1$ and defining

$$\eta(N, \|x_1-x^e\|) := \alpha_\ell^{-1}(d\alpha_u(\gamma_x(\|x_1-x^e\|)+\gamma_u(\|x_1-x^e\|\})) + d\alpha_u(\gamma_x(\varepsilon)+\gamma_u(\varepsilon)))$$

finally shows the assertion. ∎

Remark 6.3 Under the assumptions of Lemma 6.2 we can conclude that there exists $\sigma' \in \mathcal{L}_\mathbb{N}$ such that the assertion of Theorem 5.6 holds with $P(N) = \lceil N/2\rceil$, where $\lceil r\rceil$ for $r \in \mathbb{R}$ denotes the smallest integer m with $m \ge r$. Indeed, following the proof of Lemma 6.2 and choosing $\sigma(N) = \alpha_\ell^{-1}(2(\delta+C)/N)$ yields $Q_{\sigma(N)} \ge N/2$ such that we can ensure that $\|x_{u^*_{N,x}}(k,x)-x^e\| \le \sigma(N)$ holds for some $k \le N/2$. Applying then Lemma 6.2 with $x_1 = x_{u^*_{N,x}}(k,x)$ and $N' = N-k$ yields $k + P(N') \ge k + \lceil (N-k)/2\rceil \ge \lceil N/2\rceil$ which implies

$$\|x_{u^*_{N,x}}(\lceil N/2\rceil, x) - x^e\| \le \eta(N,\sigma(N)) =: \sigma'(N).$$

In fact, in all our examples numerical simulations show that the distance $\|x_{u^*_{N,x}}(k,x)-x^e\|$ becomes minimal for $k = \lceil N/2\rceil$. □

Using Lemma 6.2 we can now prove the following main theorem of this section which gives a sufficient condition for Theorem 4.2(b).

Theorem 6.4 Let the assumptions of Theorem 5.4 as well as Assumption 6.1 and (6.1) hold. Then Condition (b) of Theorem 4.2 holds for $\mathbb{Y} = \{x^e\}$.

Proof: We choose $N_0 \in \mathbb{N}$ so large and $\bar{\delta} > 0$ so small that $N_0 \geq N_1$ and $\bar{\delta} \leq \delta_c$ for the values from Lemma 6.2 and Assumption 6.1 holds and such that $\eta(N,r) < \delta_c$ in Lemma 6.2 holds for all $N \geq N_0$ and $r \in (0,\bar{\delta})$. We show the desired inequality in Condition (b) for all $x, y \in \mathcal{B}_{\bar{\delta}}(x^e)$ which particularly implies the assertion for $y \in \mathbb{Y}$, i.e., $y = x^e$.

By Lemma 6.2 and since $\eta(N,\bar{\delta}) \leq \delta_c$ and $P(N) \geq d$ (cf. the construction in the proof of Lemma 6.2) we know that the optimal trajectory starting in x satisfies

$$x_{u^*_{N,x}}(k,x) \in \mathcal{B}_{\delta_c}(x^e)$$

for $k = 0,\dots,d$. Thus, we can apply Assumption 6.1 with this x, $x_1 = y$, $u_1 = u^*_{N,x}$ and $x_2 = x_{u^*_{N,x}}(d,x)$ in order to conclude that there exists $u_2 \in \mathbb{U}^d(y)$ such that $x_{u_2}(d,y) = x_{u^*_{N,x}}(d,x)$ and $\ell(x_{u_2}(k,y),u_2(k)) \leq \ell(x_{u^*_{N,x}}(k,x),u^*_{N,x}(k)) \leq \gamma_c(\|y-x\|)$ (note that $\|x_2 - x_{u_1}(d,x)\| = 0$ by choice of x_2). Extending the control sequence u_2 by setting $u_2(k) = u^*_{N,x}(k)$ for $d,\dots,N-1$ then yields

$$\begin{aligned} V_N(y) &\leq J_N(y,u_2) \;=\; \frac{1}{N}\sum_{k=0}^{N-1} \ell(x_{u_2}(k,y),u_2(y)) \\ &\leq \frac{1}{N}\sum_{k=0}^{N-1} \ell(x_{u^*N,x}(k,x),u^*N,x(x)) + \frac{d}{N}\gamma_c(\|y-x\|) \\ &= V_N(x) + \frac{d}{N}\gamma_c(\|y-x\|). \end{aligned}$$

Setting $\gamma_V(r) = d\gamma_c(\|y-x\|)$ we thus obtain

$$V_N(y) \leq V_N(x) + \gamma_V(\|x-y\|)/N$$

and by exchanging x and y we get the converse inequality which shows Condition (b) of Theorem 4.2. □

Since we have already verified throughout this and the preceding section that all examples from Section 3 satisfy the assumptions of Theorem 5.6, this shows that Condition (c) of Theorem 4.2 holds for all these examples. Together with the results from the previous section and the observation that Condition (a) of Theorem 4.2 holds because of continuity, this shows that for all examples Theorem 4.2 can be applied with $\mathbb{Y} = \{x^e\}$ and $\ell_0 = \ell(x^e,u^e)$. Furthermore, since Corollary 5.3 applies for our examples we obtain the optimal asymptotic estimate $\lim_{N\to\infty} J_\infty(x,\mu_N) = V_\infty(x)$ for all $x \in \mathbb{X}$.

Moreover, one can check that for our examples the functions γ_f, γ_ℓ and γ_V in Conditions (a) and (b) of Theorem 4.2 are actually polynomials. Together with the (numerically observed) fact that $\sigma(N)$ from Theorem 5.6 decays exponentially in N, cf. the discussion after the proof of Theorem 5.6, this yields that the error term $\varepsilon(N-1)$ constructed in the proof of Theorem 4.2 decays exponentially in N, too.

7 An example of an optimal periodic solution

In this section we present an example which illustrates that — as discussed in Remark 5.9 — our conditions also apply to periodic optimal solutions if the stage cost is constant along these solutions.

Example 7.1 Consider the two dimensional control system with $x = (x_1, x_2)^T \in \mathbb{R}^2$ and $u = (u_1, u_2)^T \in \mathbb{R}^2$ given by

$$x(k+1) = A(u_2(k))(2x(k) + u_1(k)x(k)/\|x(k)\|),$$

for $x(k) \neq 0$ and $x(k+1) = 0$ for $x(k) = 0$, where

$$A(u_2) = \begin{pmatrix} \cos u_2 & \sin u_2 \\ -\sin u_2 & \cos u_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

and $\|\cdot\|$ is the Euclidean norm. We choose the admissible set as the ring $\mathbb{X} = \{x \in \mathbb{R}^2 \,|\, 3/4 \leq \|x\| \leq 2\}$, the control value set as $U = [-5, 5] \times [-1, 1]$ and the stage cost as $\ell(x, u) = (u_1 + 1)^2 + (u_2 - 0.1)^2$. With this cost function, one easily sees that it is optimal to first steer the system to the circle $S = \{x \in \mathbb{R}^2 \,|\, \|x\| = 1\}$ and then use the control $u^* = (-1, 0.1)^T$. Indeed, since $f(x, u^*) \in S$ and $\ell(x, u^*) = 0$ for all $x \in \mathbb{X}$, using u^* we stay on S with stage cost 0 and thus for any control sequence u_x which first steers the system from $x \in \mathbb{X}$ to S in finitely many steps and then uses the control $u_x(k) = u^*$ we get $J_\infty(x, u_x) = 0$. Since $\ell \geq 0$, this is obviously the optimal value. Since $u_2^* = 0.1$ and thus $A(u_2^*) \neq \mathrm{Id}$, the corresponding optimal trajectory is not an equilibrium but a periodic orbit.

Figure 7.1(left) shows the resulting receding horizon closed loop trajectories for $N = 4, 6, 8$ and initial values $x_0 = (0, 2)^T$ (outer trajectories) and $x_0 = (0, 3/4)^T$ (inner trajectories), respectively. The corresponding averaged infinite horizon closed loop costs $J_\infty(x_0, \mu_N)$ for $x_0 = (0, 2)^T$ and $N = 2, \ldots, 14$ are plotted in Figure 7.1(right) on a logarithmic scale.

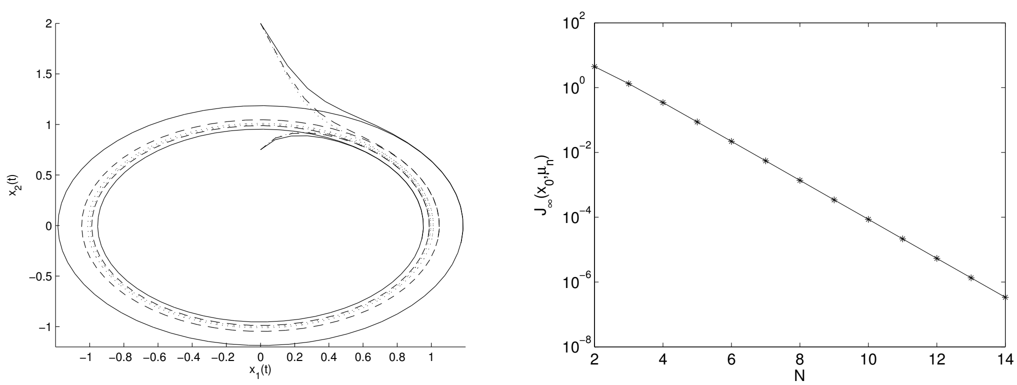

Figure 7.1: left: $x_{\mu_N}(k, x)$ for $N = 4$ (solid), $N = 6$ dashed and $N = 8$ (dotted) for $x_0 = (0, 2)^T$ (outer trajectories) and $x_0 = (0, 3/4)^T$ (inner trajectories)
right: infinite horizon closed loop costs $J_\infty(x_0, \mu_N)$ for $x_0 = (0, 2)^T$ and $N = 2, \ldots, 14$

As we see, the resulting limit cycle depends on the initial value and its radius is > 1 for $x_0 = (0, 2)^T$, < 1 for $x_0 = (0, 3/4)^T$ and converges to 1 in both cases for increasing N. Furthermore, in both cases for increasing N the solutions improve and the infinite horizon closed loop costs approach the optimal value $V_\infty(x_0) = 0$ exponentially fast. We expect that a formal proof of this property can be obtained by a generalization of the arguments in Sections 5 and 6 to periodic orbits along which ℓ constant. For the sake of brevity, however, details will be postponed to a future paper.

Again, it is interesting to look at the open loop predictions for the different initial values which are depicted in Figure 7.2 for $N = 4$ and $x_0 = (0, 2)^T$ and $x_0 = (0, 3/4)$, respectively. As in Figure 3.3, the optimal open loop solutions approach the boundary of the admissible set $\mathbb{X}$ but now it depends on the initial value whether the "outer" boundary $\|x\| = 2$ or the "inner" boundary $\|x\| = 3/4$ is approached.

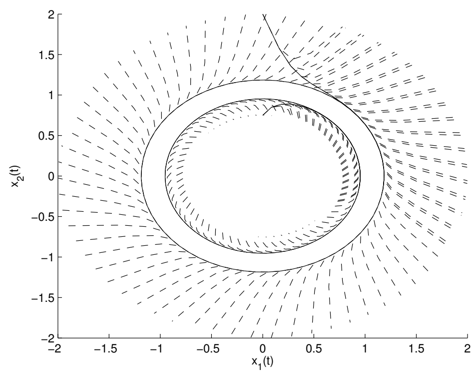

Figure 7.2: Optimal predictions $x_u(k, x_{\mu_N}(k))$ (dashed) within the receding horizon optimization for $N = 4$ with $x_0 = (0, 2)^T$ (outer trajectories) and $x_0 = (0, 3/4)^T$ (inner trajectories)

□

8 Convergence results

In our results so far we have developed conditions under which bounds for the values $J_\infty(x, \mu_N)$ and $J_N(x, \mu_N)$ along the closed loop trajectories can be deduced. In this section, we investigate the limiting behavior of the closed loop solutions. In all examples discussed so far we have seen that these solutions converge to a neighborhood of the optimal steady state or periodic solution. In this section we present a rigorous explanation for this fact. As an important consequence of this analysis, we moreover show that this convergence behavior also implies that the receding horizon closed loop solutions exhibit approximately optimal transient behavior in a finite time averaged sense. The results in this section are formulated for general sets $\mathbb{Y}$ and for this purpose we will generalize some of the properties from the Sections 5 and 6.

We start our analysis with the following theorem.

Theorem 8.1 (i) Assume there are $N_1 \in \mathbb{N}$ and $\delta \in \mathcal{L}_\mathbb{N}$ such that the inequality

$$J_K(x, \mu_N) \leq \frac{N}{K} V_N(x) - \frac{N}{K} V_N(x_{\mu_N}(K)) + \ell_0 + \delta(N)/\min\{N, K\} \tag{8.1}$$

holds for all $x \in \mathbb{X}$, all $N \geq N_1 + 1$ and all $K \in \mathbb{N}$.

Assume furthermore that there exists a set $\mathbb{Y} \subset \mathbb{X}$ and a function $\eta \in \mathcal{L}_\mathbb{N}$ such that for all $N \geq N_1$ the inequality

$$V_N(x) \geq \ell_0 + \frac{1}{N}\alpha(|x|_\mathbb{Y}) \text{ for all } x \in \mathbb{X} \setminus \mathbb{Y} \text{ with } |x|_\mathbb{Y} > \eta(N) \tag{8.2}$$

holds for some $\alpha \in \mathcal{K}_\infty$. Then for all $N \geq N_1 + 1$ the inequality

$$|x_{\mu_N}(k)|_\mathbb{Y} \leq \max\{\eta(N),\, \alpha^{-1}(\delta(N))\} \tag{8.3}$$

holds for all $k \geq N$.

(ii) If, moreover, for all $N \geq N_1$ the inequality

$$V_N(x) \leq \ell_0 + \frac{1}{N}\overline{\alpha}(|x|_\mathbb{Y}) \tag{8.4}$$

holds for some $\overline{\alpha} \in \mathcal{K}_\infty$, then for all $N \geq N_1 + 1$ the inequality

$$|x_{\mu_N}(k)|_\mathbb{Y} \leq \max\{\eta(N),\, \alpha^{-1}(\overline{\alpha}(|x|_\mathbb{Y}) + \delta(N))\} \tag{8.5}$$

holds for all $k \in \{1, \ldots, N-1\}$ and all $x \in \mathbb{X}$ with $V_k(x) \geq \ell_0$.

Proof: (i) We abbreviate $x(k) = x_{\mu_N}(k)$ and observe that for $k \geq N$ the identity $x(N, x(k-N)) = x(k)$ holds. Then, for all $N \geq N_1 + 1$, all $k \geq N$ and all $x(0) \in \mathbb{X}$ the inequality (8.1) applied with $x = x(k-N)$ and $K = N$ yields

$$V_N(x(k)) \leq V_N(x(k-N)) - J_N(x(k-N), \mu_N) + \ell_0 + \delta(N)/N. \tag{8.6}$$

Now by optimality of V_N we have $V_N(x(k-N)) \leq J_N(x(k-N), \mu_N)$. Thus, (8.6) yields

$$V_N(x(k)) \leq \ell_0 + \delta(N)/N.$$

Hence, we either get $|x(k)|_\mathbb{Y} \leq \eta(N)$ or

$$\alpha(|x(k)|_\mathbb{Y}) \leq N(V_N(x(k)) - \ell_0) \leq \delta(N)$$

which implies (8.3).

(ii) From (8.1) with $K = k \leq N$ we get

$$V_N(x(k)) \leq V_N(x) - \frac{k}{N}J_K(x, \mu_N) + \frac{k}{N}\ell_0 + \frac{1}{N}\delta(N).$$

Now $J_K(x, \mu_N) \geq V_K(x) \geq \ell_0$ yields

$$V_N(x(k)) \leq V_N(x) - \frac{k}{N}J_K(x, \mu_N) + \frac{k}{N}\ell_0 + \frac{1}{N}\delta(N) \leq V_N(x) + \frac{1}{N}\delta(N).$$

Hence, we either get $|x(k)|_\mathbb{Y} \leq \eta(N)$ or

$$\alpha(|x(k)|_\mathbb{Y}) \leq N(V_N(x(k)) - \ell_0) \leq N(V_N(x) - \ell_0 + \frac{1}{N}\delta(N)) \leq \overline{\alpha}(|x|_\mathbb{Y}) + \delta(N)$$

which implies (8.5). □

Note that the difference between the two parts of the theorem is that Part (i) provides a bound on the MPC closed loop trajectory $x_{\mu_N}(k)$ for $k \geq N$ while Part (ii) provides a bound for $k = 1, \ldots, N-1$. If both parts hold, then one could also construct a uniform upper bound of the form $\beta(\|x\|_{\mathbb{Y}}, k) + \varepsilon(N)$ which links the property desribed by the theorem to more standard practical asymptotic stability estimates.

Theorem 8.1 is readily applicable to Example 3.1, at least if we are willing to accept numerical evidence for verifying some of its assumptions. Indeed, recall from the discussion at the end of Section 6 that using numerical observations we could compute that the functions $\varepsilon(N-1)$ constructed in Theorem 4.2 decays exponentially for all our examples. Thus, comparing (4.5) with (8.1) one sees that we can set $\delta(N) = N(\varepsilon(N-1))$ in (4.5) which yields $\delta \in \mathcal{L}_{\mathbb{N}}$ because $\varepsilon(N)$ decays exponentially; in fact, δ decays exponentially, too. Inequality (8.2) is satisfied, too, which can be either seen using the estimates derived in [7, Example 4] or by numerical evaluation, which reveals that $V_N(x) \approx 3x^2/N$ outside a neighborhood of $x^e = 0$ which shrinks down exponentially fast to $\{x^e\}$ as $N \to \infty$. Hence, α in (8.2) can be chosen as $\alpha(r) = 3r^2$ and η can be chosen to decay exponentially. In summary, we can expect the MPC closed loop trajectories to end up in an exponentially shrinking neighborhood of $x^e = 0$, which is confirmed by the numerical simulations, cf. Figure 3.2. Finally, Part (ii) of Theorem 8.1 holds, too, because an upper bound of the form (8.4) follows from the discussion after (5.9). Numerically, one can also verify the assumptions of Theorem 8.1 for Example 7.1.

For Examples 3.2 and 3.3 Theorem 8.1 is, however, not directly applicable. Indeed, while numerical evidence shows that (8.1) holds for all our examples, the lower bound (8.2) does not hold for Examples 3.2 and 3.3. This is illustrated by the numerically computed optimal value functions in Figure 8.1. These functions have values well below ℓ_0 not only in a neighborhood of x^e (which would be tolerable in (8.2)) but for all $x \in \mathbb{X}$ with $x \geq x_e$.

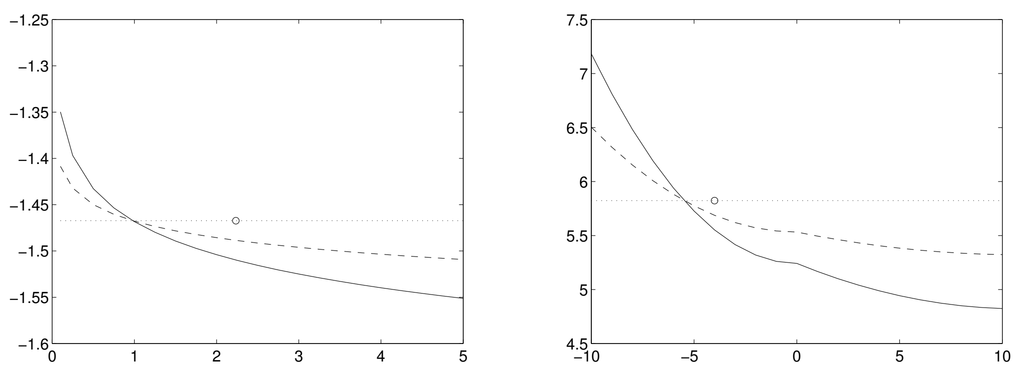

Figure 8.1: Optimal value functions V_N for Example 3.2(left) and Example 3.3(right), both for $N = 10$ (solid) and $N = 20$ (dashed). The dotted line indicates $\ell_0 = \ell(x^e, u^e)$ and the circle indicates x^e.

A remedy for this problem can be obtained once again by considering the modified stage cost as introduced in Section 5 for our examples. From (5.7) one easily concludes that the modified optimal value function satisfies $\widetilde{V}_N(x) \geq \ell(x^e, u^e) + \frac{1}{N}\alpha_\ell(\|x - x^e\|)$ and could thus be used in Theorem 8.1. The numerically computed optimal value functions in Figure 8.2 illustrate this fact and also show that these functions also satisfy (8.2) (which could also be checked rigorously using (5.9)).

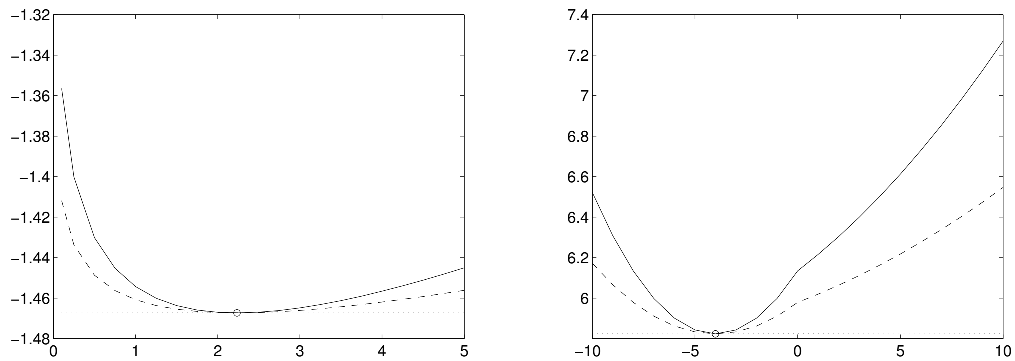

Figure 8.2: Optimal value functions $\widetilde{V}_N$ for Example 3.2(left) and Example 3.3(right) with modified stage cost $\tilde{\ell}$, both for $N = 10$ (solid) and $N = 20$ (dashed). The dotted line indicates $\ell_0 = \ell(x^e, u^e)$ and the circle indicates x^e.

Unfortunately, however, when passing from ℓ to $\tilde{\ell}$, the open loop optimal trajectories and thus also the closed loop trajectories change. For the open loop trajectories of Example 3.2 this is illustrated in Figure 8.3. Hence, we cannot simply apply Theorem 8.1 to the modified problem.

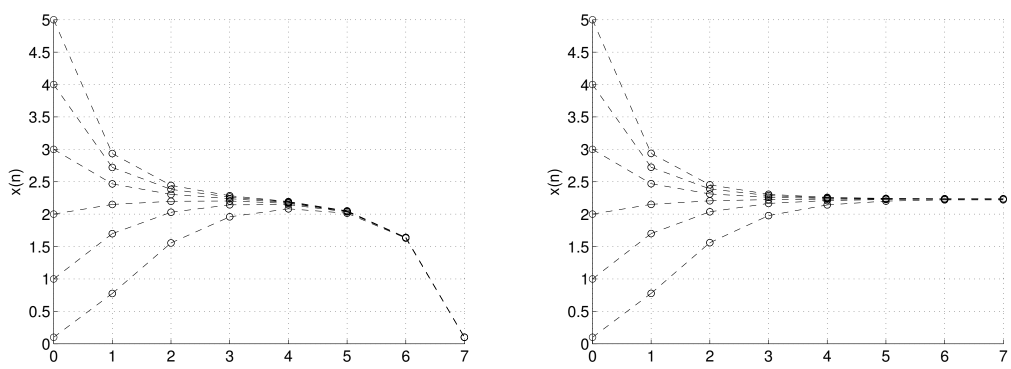

Figure 8.3: Optimal trajectories for different initial values and $N = 7$ for Example 3.2 with original stage cost ℓ(left) and modified stage cost $\tilde{\ell}$(right)

In [6] and [5, Chapter 4] terminal endpoint constraints of the type $x_u(N, x) = x_N$ are imposed in the open loop optimization. With such constraints, one can conclude from (5.3) that the functionals J_N and $\widetilde{J}_N$ only differ by the additive constant $\lambda(x) - \lambda(x_N)$ which is independent of u and consequently the optimal trajectories and thus also the closed loop trajectories coincide. Unfortunately, this argument cannot be used in our setting without terminal constraints and, in fact, the optimal trajectories differ considerably as Figure 8.3 for Example 3.2 shows.

The key to the fact that we can still use the modified problem in order to conclude convergence for the original problem lies in the fact that the optimal trajectories $x_{u^*_{N,x}}(n,x)$ and $x_{\tilde{u}^*_{N,x}}(n,x)$ (almost) coincide for small n and since the receding horizon algorithm will only pick the first element of the optimal control sequence, the receding horizon closed loop trajectories for the two problems will also be (almost) identical. This similarity between the optimal trajectories is actually no concidence but can be explained from the turnpike property. Here we give a version of these properties for optimal trajectories and general sets $\mathbb{Y}$.

Assumption 8.2 We assume that there exists $C' \geq 0$ such that for each $x \in \mathbb{X}$, the optimal control sequence $u^*_{x,N} \in \mathbb{U}^N(x)$ and each $\varepsilon > 0$ the value

$$Q_\varepsilon := \#\mathcal{P}_\varepsilon \quad \text{with} \quad \mathcal{P}_\varepsilon := \{k \in \{0,\dots,N-1\} \,|\, |x_{u^*_{N,x}}(k,x)|_Y \leq \varepsilon\}$$

satisfies the inequality $Q_\varepsilon \geq N - C'/\alpha_\ell(\varepsilon)$. □

Note that in the case $Y = \{x^e\}$ this is exactly the property Theorem 5.4 yields if we set $C' = C + \delta$ and assume the inequality $V_N(x) \leq \delta + \ell(x^e,u^e)$. As shown in Section 5, this property is satisfied for all examples from Section 3, cf. the discussion before and the proof of Theorem 5.6.

In what follows we will frequently apply Assumption 8.2 simultaneously for $p \in \mathbb{N}$ optimal trajectories with C' chosen to be the maximum of the individual constants. Denoting the respective sets by $\mathcal{P}^1_\varepsilon,\dots,\mathcal{P}^p_\varepsilon$ we will then need that for a given N_0 the intersection

$$\mathcal{P}'_\varepsilon := \mathcal{P}^1_\varepsilon \cap \ldots \cap \mathcal{P}^p_\varepsilon \cap \{0,\dots,N-N_0\}$$

contains at least one time instant P. A straightforward induction yields that if each set $\mathcal{P}^i_\varepsilon$ contains at least K elements, then the intersection contains at least $pK - (p-1)N - N_0$ elements. Hence, in order to ensure that the intersection $\mathcal{P}'_\varepsilon$ contains at least $m \in \mathbb{N}$ elements we need that each $\mathcal{P}^i_\varepsilon$ contains at least $K \geq ((p-1)N + N_0 + m)/p$ elements. This is guaranteed for the choice

$$\varepsilon \geq \alpha_\ell^{-1}\left(\frac{pC'}{N - N_0 - m}\right). \tag{8.7}$$

Note that for fixed N_0, p and m the right hand side of (8.7) tends to 0 as $N \to \infty$.

Alternatively, we could use the bound $\varepsilon \geq \sigma'(N)$ with σ' from Remark 6.3 which guarantees that P can always be chosen as $P = \lceil N/2 \rceil$ and thus yields an ε which does not depend on p. However, since we do not necessarily want to employ the assumptions imposed in this remark, in the remainder of this section we will work with (8.7).

The following lemma shows an important consequence from Assumption 8.2.

Lemma 8.3 Consider a finite horizon optimal control problem satisfying Assumption 8.2 and Condition (b) from Theorem 4.2 for some $N_0 \in \mathbb{N}$ and $\bar{\delta} > 0$. Then for all $x \in \mathbb{X}$, $N \geq N_0$ and $\varepsilon \leq \bar{\delta}$ satisfying (8.7) with $p = 1$ and $m = 1$, the set $\mathcal{P}'_\varepsilon = \mathcal{P}_\varepsilon \cap \{0, \ldots, N - N_0\}$ is nonempty and for each $P \in \mathcal{P}'_\varepsilon$, the identity

$$V_N(x) = \frac{P}{N} J_P(x, u^*_{N,x}) + \frac{N-P}{N} V_{N-P}(y) + R_1(N, \varepsilon)$$

holds, where $u^*_{N,x} \in \mathbb{U}^N(x)$ is the optimal control for initial value x and horizon N, y is an arbitrary point in $\mathbb{Y}$ and the remainder term $R_1(N, \varepsilon)$ satisfies

$$|R_1(N, \varepsilon)| \leq \frac{1}{N} \gamma_V(\varepsilon).$$

Proof: The fact that $\mathcal{P}'_\varepsilon$ is nonempty follows from the derivation of (8.7). For the proof of the claimed identity, from the optimality principle we get

$$V_N(x) = \frac{P}{N} J_P(x, u^*_{N,x}) + \frac{N-P}{N} V_{N-P}(x_{u^*_{N,x}}(P, x)). \tag{8.8}$$

Now Assumption 8.2 and the choice of N and P imply that

$$|x_{u^*_{N,x}}(P, x)|_{\mathbb{Y}} \leq \varepsilon < \bar{\delta}$$

from which by Condition (b) of Theorem 4.2 we can conclude

$$|V_{N-P}(x_{u^*_{N,x}}(P, x)) - V_{N-P}(y)| \leq \gamma_V(\varepsilon)/(N - P). \tag{8.9}$$

Here $y \in \mathbb{Y}$ is chosen such that $\|x_{u^*_{N,x}}(P, x) - y\| \leq |x_{u^*_{N,x}}(P, x)|_{\mathbb{Y}}$ holds but since Condition (b) of Theorem 4.2 implies that V_{N-P} is constant on $\mathbb{Y}$ it can actually be chosen arbitrarily. Combining (8.8) and (8.9) then yields the assertion. $\square$

In the following lemma we make use of a set of control sequence $\overline{\mathbb{U}}^P(x, \varepsilon)$ which for any $P \in \mathbb{N}$, $x \in \mathbb{X}$ and $\varepsilon > 0$ are defined by

$$\overline{\mathbb{U}}^P(x, \varepsilon) := \{u \in \mathbb{U}^P(x) \,|\, |x_u(P, x)|_{\mathbb{Y}} \leq \varepsilon\}.$$

Lemma 8.4 Assume that the function λ is constant on $\mathbb{Y}$ and Lipschitz with constant L_λ in the ball $\mathcal{B}_{\bar{\delta}}(\mathbb{Y})$ with radius $\bar{\delta} > 0$ around $\mathbb{Y}$. Then for all $u \in \overline{\mathbb{U}}^P(x, \varepsilon)$ with $\varepsilon < \bar{\delta}$ the identity

$$\widetilde{J}_P(x, u) = J_P(x, u) + \frac{1}{P} \lambda(x) - \frac{1}{P} \lambda(y) + R_2(u, P, \varepsilon)$$

hold with $|R_2(u, P, \varepsilon)| \leq L_\lambda \varepsilon / P$.

Proof: We have

$$\widetilde{J}_P(x, u) = J_P(x, u) + \frac{1}{P} \lambda(x) - \frac{1}{P} \lambda(x_u(P, x)) = J_P(x, u) + \frac{1}{P} \lambda(x) - \frac{1}{P} \lambda(y) + R_2(u, P, \varepsilon)$$

with $|R_2(u,P,\varepsilon)| = |\lambda(y)/P - \lambda(x_u(P,x))/P| \le L_\lambda |x_u(P,x)|_{\mathbb{Y}}/P \le L_\lambda \varepsilon/P$, where $y \in \mathbb{Y}$ is chosen as in the proof of Lemma 8.3 and the Lipschitz property can be used because $|x_u(P,x)|_{\mathbb{Y}} \le \varepsilon < \bar{\delta}$ holds. □

In the following lemma we assume that Assumption 8.2 holds for both the original and the modified optimal control problem. We denote the respective sets of time instants by $\mathcal{P}_\varepsilon$ and $\widetilde{\mathcal{P}}_\varepsilon$. Note that for the examples from Section 3 it follows with the same arguments as in Section 5 that Assumption 8.2 also holds for the modified problem.

Lemma 8.5 Assume that the assumptions of Lemma 8.3 hold for some $\bar{\delta} > 0$ for both the original and the modified problem, that (8.7) holds for $p = 2$ and $m = 1$ and that the assumption of Lemma 8.4 holds with the same $\bar{\delta} > 0$. Then $\mathcal{P}'_\varepsilon := \mathcal{P}_\varepsilon \cap \widetilde{\mathcal{P}}_\varepsilon \cap \{0, \dots, N - N_0\} \neq \emptyset$ and for each $P \in \mathcal{P}'_\varepsilon$ the identity

$$\widetilde{J}_P(x, \tilde{u}^*_{N,x}) = J_P(x, u^*_{N,x}) + \frac{1}{P}\lambda(x) - \frac{1}{P}\lambda(y) + R_3(P,\varepsilon)$$

holds with remainder term bounded by

$$|R_3(P,\varepsilon)| \le \frac{4}{P}(\gamma_V(\varepsilon) + L_\lambda \varepsilon).$$

Proof: First note that the choice of ε via (8.7) guarantees that $\mathcal{P}'_\varepsilon \neq \emptyset$. We pick $P \in \mathcal{P}'_\varepsilon$ and observe that the assertion of Lemma 8.3 holds for this P for the original as well as for the modified problem.

Now for an arbitrary $u \in \overline{\mathbb{U}}^P(x,\varepsilon)$ with $\varepsilon < \bar{\delta}$ we define a new control sequence $u_1 \in \overline{\mathbb{U}}^N(x)$ by setting

$$u_1(k) = \begin{cases} u(k), & k = 0, \dots, P-1 \\ u^*_{N-P, x_u(P,x)}(k-P), & k = P, \dots, N-1. \end{cases}$$

With the same arguments as in the proof of Lemma 8.3 one sees that the identity

$$J_N(x, u_1) = \frac{P}{N} J_P(x,u) + \frac{N-P}{P} V_{N-P}(y) + R_4(u, N, \varepsilon)$$

holds with $|R_4(u, N\varepsilon)| \le \gamma_V(\varepsilon)/N$. This implies

$$\frac{P}{N} J_P(x,u) \ge V_N(x) - \frac{N-P}{P} V_{N-P}(y) - R_4(u,N,\varepsilon).$$

Combining this estimate with the one from Lemma 8.3 yields

$$\frac{P}{N} J_P(x, u^*_{N,x}) = \inf_{u \in \overline{\mathbb{U}}^P(x,\varepsilon)} \frac{P}{N} J_P(x,u) + R_1(N,\varepsilon) + R_5(N,\varepsilon), \tag{8.10}$$

where $|R_5(N,\varepsilon)| \le \sup_{u \in \overline{\mathbb{U}}^P(x,\varepsilon)} |R_4(u,P,\varepsilon)|$. Using Lemma 8.4 we thus obtain

$$\frac{P}{N} \widetilde{J}_P(x, u^*_{N,x}) = \inf_{u \in \overline{\mathbb{U}}^P(x)} \frac{P}{N} \widetilde{J}_P(x,u) + R_1(N,\varepsilon) + R_5(N,\varepsilon) + R_6(N,\varepsilon)$$

where $|R_6(N,\varepsilon)| \le 2P \sup_{u\in\overline{\mathbb{U}}^P(x,\varepsilon)} |R_2(u,P,\varepsilon)|/N$ for R_2 from Lemma 8.4. Using (8.10) for the modified problem yields

$$\frac{P}{N}\widetilde{J}_P(x,\tilde{u}^*_{N,x}) = \inf_{u\in\overline{\mathbb{U}}^P(x,r)} \frac{P}{N}\widetilde{J}_P(x,u) + \widetilde{R}_1(N,\varepsilon) + \widetilde{R}_5(N,\varepsilon).$$

Combining the last two inequalities we obtain $\frac{P}{N}\widetilde{J}_P(x,\tilde{u}^*_{N,x}) = \frac{P}{N}\widetilde{J}_P(x,u^*_{N,x}) - R_1(N,\varepsilon) - R_5(N,\varepsilon) - R_6(N,\varepsilon) + \widetilde{R}_1(N,\varepsilon) + \widetilde{R}_5(N,\varepsilon)$. Now the assertion follows by applying Lemma 8.4 once more and R_3 is obtained by adding the individual remainder terms used in the proof multiplied by N/P. □

With the help of these lemmas we can now extend the convergence theorem to the modified optimal control problem.

Theorem 8.6 Consider the receding horizon scheme obtained from the optimal control problems with original cost ℓ. Assume that the problem with original cost ℓ and the problem with modified cost $\tilde{\ell}$ satisfy Assumption 8.2 and Condition (b) from Theorem 4.2 for some $N_0 \in \mathbb{N}$ and $\bar{\delta} > 0$. Assume furthermore that λ is constant on $\mathbb{Y}$ and Lipschitz with constant L_λ in the ball $\mathcal{B}_{\bar{\delta}}(\mathbb{Y})$ with radius $\bar{\delta} > 0$ around $\mathbb{Y}$. Let N_1 be such that (8.7) holds for $\varepsilon = \bar{\delta}$, $p = 6$ and $m = 1$ for all $N \ge N_1$.

(i) Assume that (8.1) holds for the receding horizon problem with original stage cost ℓ and that the optimal value functions $\widetilde{V}_N$ of the modified problem satisfy (8.2).

Then there exists $\tilde{\delta} \in \mathcal{L}_{\mathbb{N}}$ such that for all $N \ge N_1 + 1$ the inequality

$$|x_{\mu_N}(k)|_{\mathbb{Y}} \le \max\{\eta(N),\, \alpha^{-1}(\tilde{\delta}(N))\} \tag{8.11}$$

holds for all $k \ge N$.

(ii) If, moreover, $\widetilde{V}_N$ satisfies (8.4), then the inequality

$$|x_{\mu_N}(k)|_{\mathbb{Y}} \le \max\{\eta(N),\, \alpha^{-1}(\overline{\alpha}(|x|_{\mathbb{Y}}) + \tilde{\delta}(N))\} \tag{8.12}$$

holds for all $N \ge N_0 + 1$, all $k \in \{1,\dots,N-1\}$ and all $x \in \mathbb{X}$ with $\widetilde{V}_k(x) \ge \ell_0$.

Proof: It is sufficient to show that there exists $\tilde{\delta} \in \mathcal{L}_{\mathbb{N}}$ such that (8.1) holds for the modified functional and optimal value functions, i.e., that

$$\widetilde{J}_K(x,\mu_N) \le \frac{N}{K}\widetilde{V}_N(x) - \frac{N}{K}\widetilde{V}_N(x_{\mu_N}(K)) + \ell_0 + \frac{\tilde{\delta}(N)}{\min\{N,K\}} \tag{8.13}$$

holds, because from this inequality the desired estimates can be concluded just as in the proof of Theorem 8.1. Note that μ_N in (8.13) is still the receding horizon feedback law generated by the original optimal control problem with stage cost ℓ, because we want to establish convergence for the original receding horizon closed loop.

We deduce (8.13) from (8.1) and three identities obtained from the preceding lemmas. To this end, we will invoke Lemma 8.3 four times and Lemma 8.5 twice. Hence, we use Assumption 8.2 eight times and thus for $\varepsilon \le \bar{\delta}$ we choose $N \in \mathbb{N}$ such that (8.7) holds for

$p = 8$ and $m = 1$. This ensures that we can find a common $P \in \mathcal{P}'_\varepsilon$ for all six trajectories under consideration which we will use in what follows.

The first identity we will use follows from applying Lemma 8.3 at the points x and $x_{\mu_N}(K)$ which yields

$$\begin{aligned} & V_N(x) - V_N(x_{\mu_N}(K)) \\ = \; & \frac{P}{N} J_P(x, u^*_{N,x}) + \frac{N-P}{N} V_{N-P}(y) + R_1^1(N,\varepsilon) \\ - \; & \frac{P}{N} J_P(x_{\mu_N}(K), u^*_{N,x_{\mu_N}(K)}) - \frac{N-P}{N} V_{N-P}(y) - R_1^2(N,\varepsilon) \\ = \; & \frac{P}{N} J_P(x, u^*_{N,x}) - \frac{P}{N} J_P(x_{\mu_N}(K), u^*_{N,x_{\mu_N}(K)}) + R_1^1(N,\varepsilon) - R_1^2(N,\varepsilon) \end{aligned} \tag{8.14}$$

with $|R_1^i(N,\varepsilon)| \le \frac{1}{N}\gamma_V(\varepsilon)$. Proceding analogously for the modified problem we analogously obtain the second identity

$$\begin{aligned} & \widetilde{V}_N(x) - \widetilde{V}_N(x_{\mu_N}(K)) \\ = \; & \frac{P}{N} \widetilde{J}_P(x, \tilde{u}^*_{N,x}) - \frac{P}{N} \widetilde{J}_P(x_{\mu_N}(K), \tilde{u}^*_{N,x_{\mu_N}(K)}) + \widetilde{R}_1^1(N,\varepsilon) - \widetilde{R}_1^2(N,\varepsilon) \end{aligned}$$

with remainder terms satisfying the same bound. The third equation we need is obtained by applying Lemma 8.5 at the points x and $x_{\mu_N}(K)$ which yields

$$\begin{aligned} J_P(x, u^*_{N,x}) - J_P(x_{\mu_N}(K), u^*_{N,x_{\mu_N}(K)}) \; = \; & \widetilde{J}_P(x, \tilde{u}^*_{N,x}) - \widetilde{J}_P(x_{\mu_N}(K), \tilde{u}^*_{N,x_{\mu_N}(K)}) \\ & - \frac{1}{P}\lambda(x) + \frac{1}{P}\lambda(x_{\mu_N}(K)) + R_3^2(P,\varepsilon) - R_3^1(P,\varepsilon) \end{aligned}$$

with $|R_3^i(P,\varepsilon)|$ bounded as specified in Lemma 8.5. Using first (5.3), then (8.1) and then the first identity followed by the third and the second we obtain

$$\begin{aligned} & \widetilde{J}_K(x, \mu_N) - \ell_0 \\ = \; & J_K(x,\mu_N) + \frac{1}{K}\lambda(x) - \frac{1}{K}\lambda(x_{\mu_N}(K)) - \ell_0 \\ \le \; & \frac{N}{K} V_N(x) - \frac{N}{K} V_N(x_{\mu_N}(K)) + \frac{\delta(N)}{\min\{N,K\}} + \frac{1}{K}\lambda(x) - \frac{1}{K}\lambda(x_{\mu_N}(K)) \\ = \; & \frac{P}{K} J_P(x, u^*_{N,x}) - \frac{P}{K} J_P(x_{\mu_N}(K), u^*_{N,x_{\mu_N}(K)}) + \frac{N}{K} R_1^1(N,\varepsilon) - \frac{N}{K} R_1^2(N,\varepsilon) \\ + \; & \frac{\delta(N)}{\min\{N,K\}} + \frac{1}{K}\lambda(x) - \frac{1}{K}\lambda(x_{\mu_N}(K)) \\ = \; & \frac{P}{K} \widetilde{J}_P(x, \tilde{u}^*_{N,x}) - \frac{P}{K} \widetilde{J}_P(x_{\mu_N}(K), \tilde{u}^*_{N,x_{\mu_N}(K)}) + \frac{N}{K} R_1^1(N,\varepsilon) - \frac{N}{K} R_1^2(N,\varepsilon) \\ + \; & \frac{P}{K} R_3^2(P,\varepsilon) - \frac{P}{K} R_3^1(P,\varepsilon) + \frac{\delta(N)}{\min\{N,K\}} \\ = \; & \frac{N}{K} \widetilde{V}_N(x) - \frac{N}{K} \widetilde{V}_N(x_{\mu_N}(K)) + \frac{N}{K} R_1^1(N,\varepsilon) - \frac{N}{K} R_1^2(N,\varepsilon) \\ + \; & \frac{P}{K} R_3^2(P,\varepsilon) - \frac{P}{K} R_3^1(P,\varepsilon) - \frac{N}{K} \widetilde{R}_1^1(N,\varepsilon) + \frac{N}{K} \widetilde{R}_1^2(N,\varepsilon) + \frac{\delta(N)}{\min\{N,K\}} \end{aligned}$$

An inspection of the remainder terms shows that the modulus of each of them can be bounded by a term of the form $\frac{1}{K}r(\varepsilon)$ where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. Choosing ε depending on N such that equality holds in (8.7) for $p = 8$ and $m = 1$, we obtain that $\varepsilon \to 0$ as $N \to \infty$ and the choice of N_1 guarantees that $\varepsilon(N) \le \bar\delta$ holds for all $N \ge N_1$. Hence, the estimate holds for all $N \ge N_1$ and we can bound the sum of the error terms by a term of the form $\delta_1(N)/K$ with $\delta_1(N) \in \mathcal{L}_{\mathbb{N}}$. This proves (8.13) with $\tilde\delta(N) = \delta_1(N) + \delta(N)$ and thus finishes the proof. ∎

Remark 8.7 Beyond convergence, the techniques from this section also allow to conclude that the finite horizon value of the receding horizon closed loop trajectory during the transient phase is optimal up to an error term vanishing as $N \to \infty$. Here, "transient phase" refers to the time until the trajectory reaches a prescribed neighborhood of $\mathbb{Y}$. More precisely, for all $P, K \in \mathbb{N}$ with $P \le K$ and $r > 0$ we consider the set of control sequences $\widehat{\mathbb{U}}^K(x, r, P)$ for which $|x_u(k, x)|_{\mathbb{Y}} \le r$ holds for $k = P, \ldots, K-1$, i.e., all sequences for which the corresponding trajectories end up in the neighborhood $\mathcal{B}_r(\mathbb{Y})$ after time P. We show that there exists $P \in \mathbb{N}$ with $P \to \infty$ as $N \to \infty$ such that up to an error term vanishing as $N \to \infty$ the value $J_K(x, \mu_N)$ is smaller than $J_K(x, u)$ for all $u \in \widehat{\mathbb{U}}^K(x, r, P)$.

To this end, we need the property of the optimal trajectories guaranteed by Lemma 6.2. Generalized to arbitrary sets $\mathbb{Y}$, this property demands that there exists $N_1 \in \mathbb{N}$ and $\eta : \mathbb{N} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\eta(N, r) \to 0$ if $N \to \infty$ and $r \to 0$ such that the open loop optimal trajectories with horizon $N \ge N_1$ starting in $x_1 \in \mathcal{B}_{\delta_c}(x_e)$ satisfy

$$|x_{u^*_{N,x_1}}(k, x_1)|_{\mathbb{Y}} \le \eta(N, |x_1|_{\mathbb{Y}}) \tag{8.15}$$

for all $k = 0, \ldots, P(N)$ with $P(N) \ge N/2$. Again, this property holds for all examples from Section 3 as shown in Section 6.

In order to derive the transient estimate, assume that the open loop trajectories of the original problem satisfy Assumption 8.2 and that the closed loop solution x_{μ_N} converges to a neighborhood of $\mathbb{Y}$ with radius $r(N)$ with $r \in \mathcal{L}_{\mathbb{N}}$, as ensured, e.g., by Theorem 8.1 or Theorem 8.6 if we abbreviate the right hand side of (8.3) or (8.11) by $r(N)$. Throughout the derivation we will invoke Assumption 8.2 three times (via Lemma 8.3) and we will need that $\mathcal{P}'_\varepsilon$ has at least $3(N-N_0)/4$ elements. Hence, we assume that $N > 0$ is large enough in order to ensure (8.7) for the $\varepsilon > 0$ under consideration, $p = 3$ and $m = \lceil 3(N-N_0)/4 \rceil$.

Assume that there exists $\rho \in \mathcal{K}$ such that for all sufficiently small $r > 0$ and all $x \in \mathcal{B}_r(\mathbb{Y})$ and $u \in \mathbb{U}$ with $f(x, u) \in \mathcal{B}_r(\mathbb{Y})$ the inequality

$$\ell(x, u) \ge \ell_0 - \rho(r) \tag{8.16}$$

holds. Observe that also this condition is satisfied in all our examples.

Then for each $u \in \widehat{\mathbb{U}}^K(x, r, P)$ we get the estimate

$$\frac{1}{K-P} \sum_{k=P}^{K-1} \ell(x_u(k, x), \mu_N(x_{\mu_N}(k, x))) \ge \ell_0 - \rho(r(N))$$

and thus

$$J_K(x, u) \ge \frac{P}{K} J_P(x, u) + \frac{K-P}{K}(\ell_0 - \rho(r)).$$

Since each $u \in \widehat{\mathbb{U}}^K(x, r, P)$ satisfies $u|_{\{0,\dots,P-1\}} \in \overline{\mathbb{U}}^P(x, r)$, this implies

$$\inf_{u\in\widehat{\mathbb{U}}^K(x,r,P)} J_K(x,u) \geq \frac{P}{K} \inf_{u\in\overline{\mathbb{U}}^P(x,r)} J_P(x,u) + \frac{K-P}{K}(\ell_0 - \rho(r))). \tag{8.17}$$

On the other hand, using (8.14) and setting $\delta(N) = \varepsilon(N-1)$ and $R(N,\varepsilon) = |R_1^1(N,\varepsilon)/N + R_1^2(N,\varepsilon)/N|$, (4.5) becomes

$$\begin{aligned} J_K(x,\mu_N) &\leq \frac{N}{K}V_N(x) - \frac{N}{K}V_N(x_{\mu_N}(K)) + \ell_0 + \delta(N) \\ &= \frac{P}{K}J_P(x,u^*_{N,x}) - \frac{P}{K}J_P(x_{\mu_N}(K), u^*_{N,x_{\mu_N}}(K)) + \ell_0 + \delta(N) + \frac{1}{K}R(N,\varepsilon) \end{aligned}$$

Here we choose $P \in \mathcal{P}'_\varepsilon$ such that $P \leq P(N)$ from (8.15). Such a P exists because our choice of N guarantees that $\mathcal{P}'_\varepsilon \subseteq \{0,\dots,N-N_0\}$ has at least $m = \lceil 3(N-N_0)/4 \rceil$ elements which implies $\min \mathcal{P}'_\varepsilon \leq N - N_0 - \lceil 3(N-N_0)/4 \rceil \leq 3(N-N_0)/4 \leq N/2$. Moreover, the choice of m implies that $\mathcal{P}'_\varepsilon \cap \{0,\dots,\lceil N/2 \rceil\}$ has at least $\lceil (N-N_0)/4 \rceil$ elements, thus $P \geq (N-N_0)/4$ and in particular $P \to \infty$ as $N \to \infty$.

Now for $K \geq N$, from (8.15) and (8.16) we obtain the inequality $J_P(x_{\mu_N}(K), u^*_{N,x_{\mu_N}}(K)) \geq \ell_0 - \rho(\eta(N, r(N)))$ for η from (8.15). Hence we can conclude

$$J_K(x,\mu_N) \leq \frac{P}{K}J_P(x,u^*_{N,x}) + \frac{K-P}{K}\ell_0 + \frac{P}{K}\rho(\eta(N,r(N))) + \delta(N) + \frac{1}{K}R(N,\varepsilon).$$

Finally, using (8.10) and defining ε to be minimal with (8.7) with $p = 3$ and $m = \lceil 3(N - N_0)/4 \rceil$ (which implies $\varepsilon \to 0$ as $N \to \infty$), we obtain that up to an error term vanishing as $N \to \infty$ the value $J_P(x, u^*_{N,x})$ is optimal among all trajectories which satisfy $|x_u(P,x)|_{\mathbb{Y}} \leq r(N)$. Formally, combining all error terms which tend to 0 as $N \to \infty$ in one function $\Delta \in \mathcal{L}_N$, this yields

$$J_K(x,\mu_N) \leq \inf_{u\in\overline{\mathbb{U}}^P(x,r(N))} \frac{P}{K}J_P(x,u) + \frac{K-P}{K}\ell_0 + \Delta(N).$$

Comparing this estimate with (8.17), enlarging Δ in order to include the term $\rho(r(N))$, one thus obtains

$$J_K(x,\mu_N) \leq \inf_{u\in\widehat{\mathbb{U}}^K(x,r(N),P)} J_K(x,u) + \Delta(N).$$

Thus, the value $J_K(x,\mu_N)$ is — up to an error term of order N — smaller than the value of all trajectories which end up in the neighborhood $\mathcal{B}_{r(N)}(\mathbb{Y})$ after at most P steps. □

9 Conclusions and outlook

We have derived conditions under which a receding horizon control scheme yields approximately optimal average infinite horizon performance for the resulting closed loop trajectories. The conditions can be checked by means of a turnpike property and suitable controllability properties and have been rigorously verified for a number of examples. Moreover, conditions for convergence to the optimal steady state (or more general optimal solution sets $\mathbb{Y}$) and approximately optimal averaged finite horizon performance during the transient phase could be obtained.

Future research will include the extension to discounted infinite horizon problems and the investigation of optimal periodic orbits along which the stage cost ℓ is not necessarily constant. Moreover, we plan to extend the closed loop convergence results in this paper to a Lyapunov function based stability analysis for the receding horizon closed loop.

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