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EDOARDO BALLICO, ALESSANDRA BERNARDI, MARIA VIRGINIA CATALISANO, AND LUCA CHIANTINI

Abstract. For any irreducible non-degenerate variety $X \subset \mathbb{P}^r$, we relate the dimension of the s-th secant varieties of the Segre embedding of $\mathbb{P}^k \times X$ to the dimension of the (k,s)-Grassmann secant variety $GS_X(k,s)$ of X. We also give a criterion for the s-identifiability of X.

Introduction

In 1915, in an elegant XIX-century style Italian language ([Ter15, p. 97]), A. Terracini pointed out that the defectiveness of the s-th secant varieties of a Segre product $Seg(\mathbb{P}^k \times V_d)$ between a projective space $\mathbb{P}^k$ and a Veronese surface V_d is related to the fact that the set of all $\mathbb{P}^k$'s lying in the span of s independent points of V_d has not the dimension that one can expect from an obvious count of parameters. That pioneering work (also known as the second Terracini's Lemma) waited for a century before finding a first generalization. In 2001, C. Dionisi and C. Fontanari proved that Terracini's result can be formulated by replacing Veronese surface with any irreducible non-degenerate projective variety X ([DF01, Proposition 1.3]). In analogy with the notion of defectiveness, which is set forth for secant varieties, they utilized the concept of (k,s)-*Grassmann defect* that holds for the varieties X for which the set of all $\mathbb{P}^k$'s lying in the span of s independent points of X has dimension smaller than the expected one. The Zariski closure $GS_X(k,s)$ of such a set, in the corresponding Grassmannian, is called (k,s)-*Grassmann secant variety of* X. Indeed, what they proved is the following result.

Proposition 0.1. [DF01, Proposition 1.3] *Let $X \subset \mathbb{P}^r$ be an irreducible non-degenerate projective variety of dimension n. Then X is (k,s)-defective with defect $\delta_{k,s}(X) = \delta$ if and only if $Seg(\mathbb{P}^k \times X)$ is s-defective with defect $\delta_s(Seg(\mathbb{P}^k \times X)) = \delta$.*

We will show that [DF01, Proposition 1.3] holds because of a precise relation among the dimension of $GS_X(k,s)$ and the dimension of the s-th secant variety $\sigma_s(Seg(\mathbb{P}^k \times X))$ the Segre embedding of $\mathbb{P}^k \times X$ into $\mathbb{P}^{rk+r+k}$ (see Theorem 5.1 below).

Theorem *Let $X \subset \mathbb{P}^r$ be an irreducible non-degenerate projective variety of dimension n. Set $w = \min\{k, s-1\} \leq r$. Then,*

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \dim GS_X(w,s) + (w+1)(k+1) - 1.$$

Such a theorem allows to compute the dimension of $\sigma_s(Seg(\mathbb{P}^k \times X))$, in the case of $k \geq s-1$, for any irreducible non-degenerate projective variety $X \subset \mathbb{P}^r$ (see

Theorem 5.4). Moreover, such a result and its consequences endorse Conjecture 5.5 [AB09].

The key issue for the proof of Theorem 5.1 is the existence of a rational map Φ between $\sigma_s(Seg(\mathbb{P}^k \times X))$ and $GS_X(w,s)$. Given the natural way in which Φ arises, it defies belief that its importance is limited to the proof of our Theorem 5.4. We devote the whole Section 2 to a detailed description of this map because we are persuaded that it will be a key tool for further investigations on secant varieties and applications. As an example, we show how it can be exploited to shed light on another problem whose importance is probably due to its numerous applications. Given s distinct points $P_1, \dots, P_s \in X$, fix a projective linear subspace $\Pi \subset \langle P_1, \dots, P_s \rangle$. How many more $\mathbb{P}^{s-1}$ containing Π can be found among those that are s-secants to X? When the answer to this question is: "No one besides $\langle P_1, \dots, P_s \rangle$", then Π is said to be *X-identifiable*. Moreover if the general element of $GS_X(k,s)$ is contained in a unique $\mathbb{P}^{s-1}$ s-secant to X, then we say that the *(k,s)-identifiability holds for X*. The identifiability properties are studied, for the case $k = 0$, because of their many applications (see, e.g. [Kru77], [DL06], [AMR09], [CC03], [Com02], [BC11], [CC02], [Mel09], [CO11], [KB09], [ERSS05], [CC11]). Our main contribution towards a solution of this problem follows from a direct application of the map Φ above, and is summarized in the following two theorems (see Theorems 3.1 and 3.3 respectively).

Theorem *Let $k \leq s-1 < r$. The* (k,s)-identifiability *holds for X if and only if the s-identifiability holds for $Seg(\mathbb{P}^k \times X)$.*

Theorem *Let s be an integer such that $r > sn + s - 1$ and X is not s-defective. Then for all integers $k > 0$, $k \leq s-1$ such that*

$$sn + (k+1)(s-1-k) < (k+1)(r-k)$$

the (k,s)-identifiability holds for X.

In the particular case in which the variety X itself is a standard Segre variety $Seg(\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_t))$, for certain vector spaces V_i, then the identifiability properties allow to deduce many peculiar examples on linear systems of tensors $\mathcal{E} \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_t)$ (see Section 4). We like to stress here the Example 4.6 where we show that the general linear system of dimension 3 of matrices of type 4×4 and rank $s = 6$ is not identifiable and it is computed by exactly two sets of decomposable tensors (if the rank of the general linear system of dimension 3 of matrices of type 4×4 is smaller than 6 then it is identifiable, while the general linear system of dimension 3 of matrices of type 4×4 has rank 7).

In Section 1 we introduce all the basic definitions that will be needed throughout the paper and we fix some notation. Section 2 is entirely devoted to the description of the key tool of this paper, namely the ration map $\Phi : \sigma_s(Seg(\mathbb{P}^k \times X)) \dashrightarrow GS_X(w,s)$. In Section 3 we make use of Φ to study the identifiability properties of $Seg(\mathbb{P}^k \times X)$ that will be applied in Section 4 for the particular case of linear systems of tensor. In Section 5 we prove the already quoted Theorems 5.1 and 5.4 and we show some interesting consequences of them in the particular case of X being a Segre-Veronese variety.

1. Preliminaries, Notation and Basic Definitions

Throughout this paper we will always work over an algebraically closed field K of characteristic 0. All the definitions that we give in this section olds for any irreducible non-degenerate projective variety Y contained in $\mathbb{P}^m$.

Let us recall the classical definition of *secant varieties* and the more modern concept of *Grassmann secant varieties*.

Definition 1.1. The s-th *higher secant variety* $\sigma_s(Y)$ of Y, is the Zariski closure of the union of all projective linear spaces spanned by s distinct points of Y:

$$\sigma_s(Y) := \overline{\bigcup_{P_1,\dots,P_s\in Y} \langle P_1,\dots,P_s\rangle} \subset \mathbb{P}^m.$$

The expected dimension of $\sigma_s(Y)$ is

$$\text{(1)} \qquad exp\dim\sigma_s(Y) := \min\{s(\dim Y + 1) - 1; m\}.$$

When $\sigma_s(Y)$ does not have the expected dimension, Y is said to be s-*defective*, and the positive integer

$$\delta_s(Y) := exp\dim\sigma_s(Y) - \dim\sigma_s(Y)$$

is called the s-*defect* of Y.

The fact that $\sigma_s(Y)$ can have dimension smaller than the expected one, is clearly explained by the well known Terracini's Lemma (the first one). We remark here a consequence that arises when interpreting Terracini's Lemma in terms of fat points (see [CGG11, Section 2]).

Remark 1.2. Let $P_1,\dots,P_s \in Y \subset \mathbb{P}^m$ be generic distinct points and consider the 0-dimensional scheme of s 2-*fat points* $Z \subset Y$ defined by the ideal sheaf $\mathcal{I}_Z = \mathcal{I}^2_{P_1} \cap \cdots \cap \mathcal{I}^2_{P_s} \subset \mathcal{O}_Y$.

i) If $H(Z,1) = m+1$, then $\dim\sigma_t(Y) = m$ for all $t \geq s$.

ii) If $H(Z,1) = s(\dim Y + 1)$, then $\dim\sigma_t(Y) = t(\dim Y + 1) - 1$ for all $t \leq s$.

Definition 1.3. Let $0 \leq k \leq s-1 < m$ be integers and let $\mathbb{G}(k,m)$ be the Grassmannian of linear k-spaces contained in $\mathbb{P}^m$.

The (k,s)-*Grassmann secant variety of* Y, denoted with $GS_Y(k,s)$, is the closure in $\mathbb{G}(k,m)$ of the set

$$\{\Lambda \in \mathbb{G}(k,m) | \Lambda \text{ lies in the linear span of } s \text{ independent points of } Y\}.$$

Notice that, for $k = 0$, the Grassmann secant variety $GS_Y(k,s)$ coincides with the secant variety $\sigma_s(Y)$.

The expected dimension of $GS_Y(k,s)$ is the following (see eg. [CC08]):

$$\text{(2)} \qquad exp\dim GS_Y(k,s) = \min\{s(\dim Y) + (k+1)(s-1-k);\ (k+1)(m-k)\}.$$

In analogy with the theory of classical secant varieties, we define the (k,s)-*defect* of Y as the integer:

$$\delta_{k,s}(Y) := exp\dim GS_Y(k,s) - \dim GS_Y(k,s).$$

We end this section by introducing the concept of *identifiability* which will be the core of Sections 3 and 4.

Definition 1.4. Fix a linear subspace $\Pi \subset \mathbb{P}^m$ (possibly a point) and let $P_1, \dots, P_s \in Y$ be distinct points. We say that Π is *computed by* $P_1, \dots, P_s \in Y$ if Π belongs to the linear span of the points P_i's.

In this case, we say that $P_1, \dots, P_s$ provide a *decomposition* of Π.

The minimum integer s for which there exist s distinct points $P_1, \dots, P_s \in Y$ such that Π is computed by $P_1, \dots, P_s$, is called the *Y-rank of* Π. We indicate it with $r_Y(\Pi)$.

Definition 1.5. Let Y and Π be as in Definition 1.4 and let s be the Y-rank of Π. We say that Π is *Y-identifiable* if there is a unique set of distinct points $\{P_1, \dots, P_s\} \subset Y$ whose span contains Π.

Definition 1.6. Let $Y \subset \mathbb{P}^m$ as above. We say that the *(k,s)-identifiability* holds for Y if the general element of $GS_Y(k,s)$ has Y-rank equal to s and it is Y-identifiable.

When $k = 0$, we will often omit k and we will simply say that the *s-identifiability* holds for Y.

2. The map Φ

From now on, with X we will always denote an irreducible non-degenerate projective variety of dimension n contained in $\mathbb{P}^r$. For any integer $k \geq 0$, set $N = rk+r+k$ and let $\varphi : \mathbb{P}^k \times X \to \mathbb{P}^N$ be the Segre embedding of $\mathbb{P}^k \times X$. The image of φ is the Segre variety $Seg(\mathbb{P}^k \times X) \subset \mathbb{P}^N$.

The aim of this section is to study a projective rational map $\Phi = \Phi(X,k,s)$ from the s-th secant variety $\sigma_s(\mathbb{P}^k \times X)$ of the Segre variety $Seg(\mathbb{P}^k \times X)$ into the Grassmann secant variety $GS_X(k,s)$.

We will give a definition of the morphism, in terms of local coordinates. Then, we will show how it allows to link the main secant properties of $Seg(\mathbb{P}^k \times X)$ with the Grassmann-secant properties of X.

Notation 2.1. For any choice of t points

$$A_i = (a_{i,0}, \dots, a_{i,r}) \in K^{r+1}, \quad 1 \leq i \leq t,$$

we denote by $(A_1, \dots, A_t)$ the $t(r+1)$-uple

$$(a_{1,0}, \dots, a_{1,r}, \dots, a_{t,0}, \dots, a_{t,r}).$$

Let $(\lambda_0, \dots, \lambda_k)$ and $(x_0, \dots, x_r)$ be sets of homogeneous coordinates for the points $\Lambda \in \mathbb{P}^k$ and $P \in X$, respectively.

Consider the point $\varphi(\Lambda, P) \in Seg(\mathbb{P}^k \times X)$, so that, in coordinates:

$$\varphi(\Lambda, P) = (\lambda_0 x_0, \dots, \lambda_0 x_r, \lambda_1 x_0, \dots, \lambda_1 x_r, \dots, \lambda_k x_0, \dots, \lambda_k x_r).$$

Accordingly with the previous notation, we have:

$$\varphi(\Lambda, P) = (\lambda_0 P, \dots, \lambda_k P).$$

Let A be a general point in $\sigma_s(Seg(\mathbb{P}^k \times X))$. Then there exist s distinct points $\Lambda_1, \dots, \Lambda_s \in \mathbb{P}^k$ and s distinct points $P_1, \dots, P_s \in X$ such that $A \in \langle \varphi(\Lambda_1, P_1), \dots, \varphi(\Lambda_s, P_s) \rangle$.

Choose a set of homogeneous coordinates $(a_0, \dots, a_N)$ for A. By a suitable choice of the homogeneous coordinates $(\lambda_{i,0}, \dots, \lambda_{i,k})$ of the points Λ_i, we can write:

$$A = (a_0, \dots, a_N) = \varphi(\Lambda_1, P_1) + \cdots + \varphi(\Lambda_s, P_s) = \\ = (\lambda_{1,0}P_1, \dots, \lambda_{1,k}P_1) + \cdots + (\lambda_{s,0}P_s \dots, \lambda_{s,k}P_s).$$

In the previous notation, this amounts to write

$$A = (\lambda_{1,0}P_1 + \cdots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \cdots + \lambda_{s,k}P_s).$$

For any general point A as above, we set:

$$\Phi(A) = \langle \lambda_{1,0}P_1 + \cdots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \cdots + \lambda_{s,k}P_s \rangle. \tag{3}$$

Observe that, since A is general, then the right side of the equality represents a linear space of dimension $w = \min\{s-1, k\}$.

We want to show that, in this way, we get indeed a rational map

$$\Phi : \sigma_s(Seg(\mathbb{P}^k \times X)) \dashrightarrow GS_X(w, s).$$

The map Φ is well defined, if

$$(\alpha a_0, \dots, \alpha a_N), \quad \alpha \in K - \{0\}$$

is another set of homogeneous coordinates of A, then

$$\alpha(a_0, \dots, a_N) = \alpha(\lambda_{1,0}P_1 + \cdots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \cdots + \lambda_{s,k}P_s),$$

and in this case, obviously, $\lambda_{1,i}P_1 + \cdots + \lambda_{s,i}P_s$ and $\alpha(\lambda_{1,i}P_1 + \cdots + \lambda_{s,i}P_s)$, for $0 \le i \le k$, represent the same projective points.

Moreover, if there exist points $\mathcal{M}_i = (\mu_{i,0}, \dots, \mu_{i,k}) \in \mathbb{P}^k$ and $Q_i = (y_{i,0}, \dots, y_{i,r}) \in X$ such that

$$A = (a_0, \dots, a_N) = \varphi(\mathcal{M}_1, Q_1) + \cdots + \varphi(\mathcal{M}_s, Q_s),$$

we get the following equality of $(r+1)(k+1)$-tuples:

$$(\lambda_{1,0}P_1 + \cdots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \cdots + \lambda_{s,k}P_s) = \\ = (\mu_{1,0}Q_1 + \cdots + \mu_{s,0}Q_s, \dots, \mu_{1,k}Q_1 + \cdots + \mu_{s,k}Q_s).$$

Hence

$$\lambda_{1,i}P_1 + \cdots + \lambda_{s,i}P_s = \mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s; \quad i = 0, \dots, k. \tag{4}$$

It follows that Φ is consistent.

Next, we give a characterization of points belonging to the inverse image $\Phi^{-1}(\Pi)$ of a space $\Pi \in GS_X(k, s)$.

Lemma 2.2. *Let $w = \min\{k, s-1\} \le r$ and take a general point $\Pi \in GS_X(w, s)$. Assume $\Pi \subset \langle P_1, \dots, P_s \rangle$ for $P_1, \dots, P_s \in X$ distinct points. Let B be a general element in $\Phi^{-1}(\Pi)$. Hence there exist points $\mathcal{N}_1, \dots, \mathcal{N}_s \in \mathbb{P}^k$ such that*

$$B = \varphi(\mathcal{N}_1, P_1) + \cdots + \varphi(\mathcal{N}_s, P_s).$$

Proof. Let us stress, before beginning the proof, that the generality hypothesis on Π and B are crucial, for the argument.

By definition, we know that there are points $Q_1, \dots, Q_s \in X$ and $\mathcal{M}_1, \dots, \mathcal{M}_s \in \mathbb{P}^k$, with $\mathcal{M}_i = (\mu_{i,0}, \dots, \mu_{i,k})$ for $i = 1, \dots, s$, such that

$$B = \varphi(\mathcal{M}_1, Q_1) + \cdots + \varphi(\mathcal{M}_s, Q_s) = \\ = (\mu_{1,0}Q_1, \dots, \mu_{1,k}Q_1) + \cdots + (\mu_{s,0}Q_s \dots, \mu_{s,k}Q_s).$$

Since

$$\Phi((\mu_{1,0}Q_1, \dots, \mu_{1,k}Q_1) + \cdots + (\mu_{s,0}Q_s \dots, \mu_{s,k}Q_s)) = \\ = \langle \mu_{1,0}Q_1 + \cdots + \mu_{s,0}Q_s, \dots, \mu_{1,k}Q_1 + \cdots + \mu_{s,k}Q_s \rangle$$

and

$$\Pi = \langle \lambda_{1,0}P_1 + \cdots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \cdots + \lambda_{s,k}P_s \rangle$$

it follows that each point $\mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s$, $(i = 0, \dots, k)$, lies in the span of the points $\lambda_{1,j}P_1 + \cdots + \lambda_{s,j}P_s$, $(j = 0, \dots, k)$.

By the definition of w and by the generality of Π, we may assume that the points $\lambda_{1,j}P_1 + \cdots + \lambda_{s,j}P_s$, $(j = 0, \dots, w)$ are independent. It follows that, both for $w = s-1$ or for $w = k$, there are coefficients $\alpha_{i,j} \in K$, such that

$$\mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s = \sum_{j=0}^{w} \alpha_{i,j}(\lambda_{1,j}P_1 + \cdots + \lambda_{s,j}P_s) = \\ = \left(\sum_{j=0}^{w} \alpha_{i,j}\lambda_{1,j}\right) P_1 + \cdots + \left(\sum_{j=0}^{w} \alpha_{i,j}\lambda_{s,j}\right) P_s$$

for $i = 0, \dots, k$.

So, by setting $\nu_{h,i} = \left(\sum_{j=0}^{w} \alpha_{i,j}\lambda_{h,j}\right)$, we have

$$\mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s = \nu_{1,i}P_1 + \cdots + \nu_{s,i}P_s \ , \quad i = 0, \dots, k.$$

Hence we get:

$$\begin{aligned} B &= (\mu_{1,0}Q_1 + \cdots + \mu_{s,0}Q_s, \dots, \mu_{1,k}Q_1 + \cdots + \mu_{s,k}Q_s) = \\ &= (\nu_{1,0}P_1 + \cdots + \nu_{s,0}P_s, \dots, \nu_{1,k}P_1 + \cdots + \nu_{s,k}P_s) = \\ &= (\nu_{1,0}P_1, \dots, \nu_{1,k}P_1) + \cdots + (\nu_{s,0}P_s, \dots, \nu_{s,k}P_s) = \\ &= \varphi(\mathcal{N}_1, P_1) + \cdots + \varphi(\mathcal{N}_s, P_s), \end{aligned}$$

where $\mathcal{N}_i = (\nu_{i,0}, \dots, \nu_{i,k}) \in \mathbb{P}^k$ for all i. □

3. Some consequences on the identifiability of general points

The previous construction of the map Φ in Section 2, as well as Lemma 2.2, determine the following analogue of the main Theorem in [DF01], for identifiability.

Theorem 3.1. *Let $k \leq s-1 < r$. The (k,s)-*identifiability *holds for X if and only if the s-identifiability holds for $Seg(\mathbb{P}^k \times X)$.*

Proof. Let Π be a general element of $GS_X(k,s)$. If there exist two different sets of distinct points $\{P_1, \dots, P_s\}, \{Q_1, \dots, Q_s\} \subset X$ such that

$$\Pi \subset \langle P_1, \dots, P_s \rangle \quad \text{and} \quad \Pi \subset \langle Q_1, \dots, Q_s \rangle,$$

then, by Lemma 2.2, for a general point B in $\Phi^{-1}(\Pi)$, we have points $\mathcal{M}_i$'s and $\mathcal{N}_i$'s in $\mathbb{P}^k$, $i = 1, \dots, s$, with:

$$B = \varphi(\mathcal{M}_1, Q_1) + \cdots + \varphi(\mathcal{M}_s, Q_s) = \varphi(\mathcal{N}_1, P_1) + \cdots + \varphi(\mathcal{N}_s, P_s).$$

Since $\{P_1, \dots, P_s\} \neq \{Q_1, \dots, Q_s\}$, we get that B lies in the span of two distinct sets of points of $Seg(\mathbb{P}^k \times X)$.

Now let A be a general element of $\sigma_s(Seg(\mathbb{P}^k \times X))$. If

$$A = \varphi(\Lambda_1, P_1) + \cdots + \varphi(\Lambda_s, P_s) = \varphi(\mathcal{M}_1, Q_1) + \cdots + \varphi(\mathcal{M}_s, Q_s)$$

are two different decompositions of A, then, by the definition of Φ, $\Pi = \Phi(A)$ lies in the span of the two sets of points $\{P_1, \dots, P_s\}$ and $\{Q_1, \dots, Q_s\}$. It suffices to prove that these two sets of points are distinct.

Since A is general, we may assume that the two sets of points are both independent. Since:

$$\lambda_{1,i}P_1 + \cdots + \lambda_{s,i}P_s = \mu_{1,i}Q_1 + \cdots + \mu_{s,i}Q_s, \quad i = 0, \dots, k,$$

then $\{P_1, \dots, P_s\} = \{Q_1, \dots, Q_s\}$ implies $\lambda_{j,i} = \mu_{j,i}$ for all j, i. This contradicts the fact that the two decompositions of A are different. □

Corollary 3.2. *If the codimension of X is bigger than s, then $Seg(\mathbb{P}^{s-1} \times X)$ is s-identifiabile.*

Proof. Enough to observe that the general s-secant $(s-1)$-space cuts X only in s points, thus it is obvious that a general $(s-1)$-space contained in a s-secant $(s-1)$-space, is contained in just one of them!

Then, since under our numerical assumptions we have $r - n > s$ (hence $s-1 < r$), we may use the previous theorem to get the conclusion. □

Using a result of [BC11], which, in turn, is based on the main result of [CGG11], we are able to prove a criterion for the Grassmann identifiability.

Theorem 3.3. *Let s be an integer such that $r > sn+s-1$ and X is not s-defective. Then for all integers $k > 0$, $k \leq s-1$ such that*

$$sn + (k+1)(s-1-k) < (k+1)(r-k)$$

the (k, s)-identifiability holds for X.

Proof. Theorem 3.1 says that (k, s)-identifiability holds for X when s-identifiability holds for $Seg(\mathbb{P}^k \times X)$. Under our numerical assumptions, the s-secant variety of $Seg(\mathbb{P}^k \times X)$ cannot cover the linear span of $Seg(\mathbb{P}^k \times X)$. Thus we may apply the main theorem of [BC11], and conclude that $Seg(\mathbb{P}^k \times X)$ is s-identifiable. □

Indeed, the proof of Theorem 3.1 can be enhanced, to give the following, more precise result:

Proposition 3.4. *Let w, Π, B be as in Lemma 2.2. The following two sets:*

$$\mathcal{E}(\Pi) = \{(P_1, \dots, P_s) \in X^s \mid \Pi \subset \langle P_1, \dots, P_s \rangle\}$$

$$\mathcal{E}(B) = \{(P_1, \dots, P_s) \in X^s \mid \exists\, \mathcal{N}_1, \dots, \mathcal{N}_s \in \mathbb{P}^k \text{ with } B \in \langle (\mathcal{N}_1, P_1), \dots, (\mathcal{N}_s, P_s) \rangle\}$$

have the same cardinality.

Proof. Almost immediate, following the proof of Theorem 3.1. The unique warning is that the set $\{P_1, \dots, P_s\}$ that we use in the argument, must be independent. Since Π, B are general, this is true for *all* the elements of $\mathcal{E}(B)$ or $\mathcal{E}(\Pi)$, when these sets are finite, and for infinitely many elements, when they are infinite. □

To be even more precise, the sets $\mathcal{E}(\Pi)$ and $\mathcal{E}(B)$ can be endowed with a quasi-projective structure and Lemma 2.2 shows indeed that there exists a birational map $\mathcal{E}(\Pi) \to \mathcal{E}(B)$.

We will not explore this point of view any further, because we do not need it in the sequel.

4. Linear systems of tensors

In this section, we collect some consequences of the previous theory, trying to explain properly its range of application.

We consider a linear space V over K of tensors of type $n_1 + 1, \dots, n_t + 1$. A *linear system* of tensors is just a linear subspace of V. In the projective setting, tensors of type $n_1+1, \dots, n_t+1$, up to scalar multiplication, determine a projective space $\mathbb{P}^M$, where $M = (\Pi n_i) - 1$. A linear system of tensors is a linear subspace $\mathcal{E}$ of $\mathbb{P}^M$.

We take the *dimension* of $\mathcal{E}$ to be the *projective* dimension of the linear subspace associated to $\mathcal{E}$ (i.e. the affine dimension, minus 1).

Inside the space of tensors, there is the subvariety X of *decomposable tensors*, which corresponds to the Segre embedding $X = Seg(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}) \subset \mathbb{P}^M$.

Definition 4.1. We say that the linear system $\mathcal{E}$ of tensors is *computed* by s decomposable tensors $P_1, \dots, P_s \in X$ if for all $P \in \mathcal{E}$ there are scalars $a_1, \dots, a_s$ such that:

$$P = a_1 P_1 + \cdots + a_s P_s.$$

Geometrically, this means that the linear space associated to $\mathcal{E}$ lies in the span of the points $P_1, \dots, P_s$.

We say that $\mathcal{E}$ has *rank* s if s is the minimum such that there are s tensors in X which compute $\mathcal{E}$.

We say that a linear system $\mathcal{E}$ of rank s is *identifiable* if there exists a unique set of s decomposable tensors, that compute $\mathcal{E}$.

We say that tensors of type $n_1 + 1, \dots, n_t + 1$ are (k, s)*-identifiable* if the general linear system $\mathcal{E}$ of such tensors, of dimension k and rank s, is identifiable.

It is immediate to see that the previous terminology is consistent with the general terminology of the paper, once one consider the linear subspace associated to a linear system.

The map Φ constructed in the previous sections maps a tensor P of type $k + 1, n_1 + 1, \dots, n_t + 1$ to a linear system of dimension k of tensors of type $n_1 + 1, \dots, n_t + 1$. Roughly speaking, the map takes the tensor T to the linear space generated by its $k+1$ slices along the first direction.

Thus, all the results in the previous section apply to the identifiability of linear systems of tensors. In particular, for instance, we see that:

Remark 4.2.

(i) The general linear systems of dimension k of tensors of type $n_1+1,\dots,n_t+1$ has rank s if and only if s is the minimum such that the secant variety $\sigma_s(\mathbb{P}^k\times\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_t})$ covers the projective space $\mathbb{P}^N$, $N=(M+1)(k+1)-1$.

(ii) There are exactly q sets of decomposable tensors that compute a general linear system of tensors of type $n_1+1,\dots,n_t+1$ if and only if there are exactly q decomposable tensors that compute a general tensors of type $k+1,n_1+1,\dots,n_t+1$.

(iii) Tensors of type of type $n_1+1,\dots,n_t+1$ are (k,s)-identifiable if and only if tensors of type $k+1,n_1+1,\dots,n_t+1$ are s-identifiable.

Let us see how the previous remarks allows to translate some known facts about tensors to facts about linear systems of tensors.

Example 4.3. *For $m>4$, the general linear pencil of tensors of type $2\times\cdots\times 2$, (m-times) has rank $\lceil 2^m/(m+1)\rceil$.*

The general linear pencil as above, of rank $s\le 2^{m-1}/m$, is identifiable.

The first fact follows from the main result in [CGG11]. The second, from the main result of [BC11].

Example 4.4. *The general linear pencil of tensors of type $2\times 2\times 2\times 2$ has rank 6.*

The general linear pencil of tensors of type $2\times 2\times 2\times 2$, of rank $s<5$, is identifiable.

The general linear pencil of tensors of type $2\times 2\times 2\times 2$, of rank 5, is NOT *identifiable: it is computed by exactly two sets of decomposable tensors.*

Just use the main results in [CGG11], and Proposition 4.1 of [BC11].

There are also results for linear systems of matrices, which, as far as we know, cannot be found in the classical literature.

Example 4.5. *The general linear system of rank s and dimension $c-1$, of matrices of type $a\times b$, with $a\le b\le c$, is identifiable, as soon as $s\le ab/16$.*

It follows from the main result in [CO11].

Example 4.6. *The general linear system of dimension 3 of matrices of type 4×4 has rank 7.*

The general linear system of dimension 3 of matrices of type 4×4 and rank $s<6$ is identifiable.

The general linear system of dimension 3 of matrices of type 4×4 and rank $s=6$ is NOT *identifiable: it is computed by exactly two sets of decomposable tensors.*

Use the main results in [AOP09], and Theorem 1.3 of [CO11].

Tons of similar results, about the identifiability of linear systems of tensors, can be found by rephrasing, from the point of view of Remark 4.2 the examples that the reader can find in [Kru77], [DL06], [Lic85], [CGG11], [BC11], [AOP09], [CO11], [BCO] etc.

We will not get further on this subject.

5. Some consequences on the dimension of secant varieties of Segre varieties

The construction introduced with the map Φ in Section 2, as well as the obvious remark at the beginning of the proof of Corollary 3.2, allow us to reproduce the proof of the main result in [DF01], and also provide some new results on the defectivity of Segre varieties, which seem missing in the literature.

Let us start with a result on the relation between the dimension of $\sigma_s(Seg(\mathbb{P}^k \times X))$ and $GS_X(w,s)$.

Theorem 5.1. *Assume, as always, $w = \min\{k, s-1\} \leq r$. Then we have:*

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \dim GS_X(w,s) + (w+1)(k+1) - 1.$$

Proof. Let Π be a general element of $GS_X(w,s)$, that is, Π is a w-space contained in $\langle P_1, \dots, P_s \rangle$, where the P_i are independent points of X.

If we prove that $\dim \Phi^{-1}(\Pi) = (w+1)(k+1) - 1$, we are done.

Even if $w < k$, we can fix scalars $\lambda_{i,j} \in K$, with $i = 1, \dots, s$ and $j = 0, \dots, k$, such that

$$\Pi = \langle \lambda_{1,0}P_1 + \cdots + \lambda_{s,0}P_s, \dots, \lambda_{1,k}P_1 + \cdots + \lambda_{s,k}P_s \rangle.$$

Consider the points $\Lambda_i = (\lambda_{i,0}, \dots, \lambda_{i,k}) \in \mathbb{P}^k$, and let

$$A = \varphi(\Lambda_1, P_1) + \cdots + \varphi(\Lambda_s, P_s) \in \sigma_s(Seg(\mathbb{P}^k \times X)). \tag{5}$$

Obviously $A \in \Phi^{-1}(\Pi)$ and, for a general choice of the scalars, A will be a general point of $\Phi^{-1}(\Pi)$.

Since $s < r+1$, without loss of generality, we may assume that the P_i are coordinate points, say

$$P_1 = (1, 0, \dots, 0), P_2 = (0, 1, \dots, 0), \dots, P_s = (0, \dots, 0, 1, \dots, 0).$$

With this choice of coordinates, it is easy to see that

$$\Phi(A) = \langle (\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{s,0}, 0, \dots, 0), \dots, (\lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{s,k}, 0, \dots, 0) \rangle,$$

Now, fix another general point $B \in \Phi^{-1}(\Pi)$. By Lemma 2.2, we know that there are points $\mathcal{M}_i = (\mu_{i,0}, \dots, \mu_{i,k}) \in \mathbb{P}^k$ with

$$B = \varphi(\mathcal{M}_1, P_1) + \cdots + \varphi(\mathcal{M}_s, P_s)$$

and so:

$$\Phi(B) = \langle (\mu_{1,0}, \mu_{2,0}, \dots, \mu_{s,0}, 0, \dots, 0), \dots, (\mu_{1,k}, \mu_{2,k}, \dots, \mu_{s,k}, 0, \dots, 0) \rangle.$$

Since $\Phi(A) = \Phi(B)$, in the case $w = k \leq s-1$, it follows that each point $(\mu_{1,i}, \mu_{2,i}, \dots, \mu_{s,i}, 0, \dots, 0)$, $(i = 0, \dots, k)$, lies in the span of the $k+1$ points $(\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0)$, $(j = 0, \dots, k)$.

In case $w = s-1 < k$, each point $(\mu_{1,i}, \mu_{2,i}, \dots, \mu_{s,i}, 0, \dots, 0)$, $(i = 0, \dots, k)$, lies in the span of $w+1$ independent points among the $k+1$ points $(\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0)$, $(j = 0, \dots, k)$, and we may assume that these $w+1$ independent points are $(\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0)$, with $j = 0, \dots, w$.

In other words, there exist $(w+1)(k+1)$ elements $\alpha_{i,j} \in K$ s.t.

$$(\mu_{1,i}, \mu_{2,i}, \dots, \mu_{s,i}, 0, \dots, 0) = \sum_{j=0}^{w} \alpha_{i,j}(\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{s,j}, 0, \dots, 0),$$

where $i = 0, \dots, k$.

Equivalently, the following linear system

$$\begin{pmatrix} M & 0 & 0 & \dots & 0 & 0 \\ 0 & M & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 & M \end{pmatrix} \begin{pmatrix} \alpha_{0,0} \\ \dots \\ \alpha_{0,k} \\ \alpha_{1,0} \\ \dots \\ \alpha_{1,k} \\ \dots \\ \alpha_{k,0} \\ \dots \\ \alpha_{k,k} \end{pmatrix} = \begin{pmatrix} \mu_{1,0} \\ \dots \\ \mu_{s,0} \\ \mu_{1,1} \\ \dots \\ \mu_{s,1} \\ \dots \\ \mu_{1,k} \\ \dots \\ \mu_{s,k} \end{pmatrix}$$

where $M = \begin{pmatrix} \lambda_{1,0} & \dots & \lambda_{1,k} \\ \lambda_{2,0} & \dots & \lambda_{2,k} \\ \dots & \dots & \dots \\ \lambda_{s,0} & \dots & \lambda_{s,k} \end{pmatrix}$, has solutions. Since A is general, the rank of the coefficient matrix of this linear system is $(w+1)(k+1)$.

Now, since

$$\begin{aligned} B &= \varphi(\mathcal{M}_1, P_1) + \dots + \varphi(\mathcal{M}_s, P_s) \\ &= (\mu_{1,0}P_1, \dots, \mu_{1,k}P_1) + \dots + (\mu_{s,0}P_s \dots, \mu_{s,k}P_s) \\ &= (\mu_{1,0}, \mu_{2,0} \dots, \mu_{s,0}, 0, \dots, 0, \mu_{1,1}, \mu_{2,1} \dots, \mu_{s,1}, 0, \dots, 0, \\ &\quad \dots\dots, \mu_{1,k}, \mu_{2,k}, \dots, \mu_{s,k}, 0, \dots, 0), \end{aligned}$$

it immediately follows that the dimension of $\Phi^{-1}(\Pi)$ is $(w+1)(k+1)-1$. □

As consequence of Theorem 5.1, we get Terracini's Theorem of [DF01].

Corollary 5.2. *Let $k \leq s-1 < r$. Then X is (k,s)-defective with defect $\delta_{k,s}(X) = \delta$ if and only if $Seg(\mathbb{P}^k \times X)$ is s-defective with defect $\delta_s(Seg(\mathbb{P}^k \times X)) = \delta$.*

Proof. By a direct computation we get

$$exp\dim(\sigma_s(Seg(\mathbb{P}^k \times X))) = exp\dim(GS_X(k,s)) + k^2 + 2k.$$

By Theorem 5.1, since $w = k$, we have

$$\dim(\sigma_s(Seg(\mathbb{P}^k \times X))) = \dim(GS_X(k,s)) + k^2 + 2k.$$

Hence, recalling Definition 1.1, we get that the s-defect of $\sigma_s(Seg(\mathbb{P}^k \times X))$ and the s-defect of $GS_X(k,s)$ are the same. □

Next, we get some results about the defectivity or non-defectivity of the s-th higher secant variety of $Seg(\mathbb{P}^k \times X)$.

Lemma 5.3. *For $s-1 < k < r$, we have*

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \min\{s(k+n+1)-1; s(k+r-s+2)-1\}$$

Proof. By Theorem 5.1 we get

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \dim GS_X(s-1,s) + s(k+1) - 1.$$

It is well known, (see, for instance, [CC08, Section 2]), that the dimension of $GS_X(s-1,s)$ is the smallest between sn and the dimension of the Grassmannian $\mathbb{G}(s-1,r)$, i.e.

$$\dim GS_X(s-1,s) = \min\{sn;\ s(r-s+1)\}.$$

Hence

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \min\{sn;\ s(r-s+1)\} + s(k+1) - 1\}$$

and the conclusion follows. □

Theorem 5.4. *Let $X \subset \mathbb{P}^r$ be an irreducible non-degenerate projective variety of dimension n.*

(i) *If $s-1 \geq r$, then*

$$\sigma_s(Seg(\mathbb{P}^k \times X)) = \mathbb{P}^N,$$

so $\sigma_s(Seg(\mathbb{P}^k \times X))$ is not defective.

(ii) *Let $s-1 < \min\{r;\ k\}$;*

(a) *if $s-1 \leq r-n$, then*

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = s(k+n+1) - 1,$$

and $\sigma_s(Seg(\mathbb{P}^k \times X))$ is not defective;

(b) *if $s-1 > r-n$, then*

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = s(k+r-s+2) - 1,$$

and $\sigma_s(Seg(\mathbb{P}^k \times X))$ is defective.

(iii) *If $s-1 = k < r$, then*

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \min\{s(k+n+1) - 1\ ,\ N\},$$

and $\sigma_s(Seg(\mathbb{P}^k \times X))$ is not defective.

(iv) *If $k < s-1 < r$, then*

$$\dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \dim GS_X(k,s) + k^2 + 2k.$$

Proof. (i) It is enough to prove this case for $s-1 = r$.

Let $P_1, \dots, P_s$ be independent points in X. We may assume that they are the coordinate points

$$P_1 = (1, 0, \dots, 0), P_2 = (0, 1, \dots, 0), \dots, P_s = (0, \dots, 0, 1).$$

Now let A be a general point in $\mathbb{P}^N$, (recall that $N = (k+1)(r+1) - 1$)

$$A = (\lambda_{1,0}, \lambda_{2,0}, \dots, \lambda_{s,0}, \dots, \lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{s,k}).$$

Then

$$A = \varphi(\Lambda_1, P_1) + \cdots + \varphi(\Lambda_s, P_s)$$

and we are done.

(ii) By Lemma 5.3 we immediately get the dimensions of $\sigma_s(Seg(\mathbb{P}^k \times X))$ both in case (a) and in case (b).

Since in case (ii)(a) we have

$$N - s(k+n+1) - 1 = (k+1)(r+1) - s(k+n+1) \geq$$
$$(k+1)(s+n) - s(k+n+1) = (k+1-s)n > 0,$$

and (see Section 1)

$$exp \dim \sigma_s(Seg(\mathbb{P}^k \times X)) = \min\{s(k+n+1) - 1,\ N\}, \tag{6}$$

we get

$$exp \dim \sigma_s(Seg(\mathbb{P}^k \times X)) = s(k+n+1) - 1,$$

and so in this case $\sigma_s(Seg(\mathbb{P}^k \times X))$ is not defective.

In case (ii)(b) we have

$$s(k+n+1)-1-\dim\sigma_s(Seg(\mathbb{P}^k \times X))$$
$$= s(k+n+1)-1-(s(k+r-s+2)-1) = s(n-r+s-1) > 0$$

and

$$N - \dim\sigma_s(Seg(\mathbb{P}^k \times X)) = (k+1)(r+1)-1-(s(k+r-s+2)-1) =$$
$$= (r-s+1)(k-s+1) > 0.$$

Hence $\dim\sigma_s(Seg(\mathbb{P}^k \times X)) < exp\dim\sigma_s(Seg(\mathbb{P}^k \times X))$. It follows that $\sigma_s(Seg(\mathbb{P}^k \times X))$ is defective.

(iii) For $s-1=k$, we have $s(k+r-s+2)-1=N$, hence by Lemma 5.3 and (6) we get the conclusion.

(iv) Obvious from Theorem 5.1. □

If $k = r-n$, by applying the theorem above we get the following interesting result.

Corollary 5.5. *If $k = r-n$, then $\sigma_s(Seg(\mathbb{P}^k \times X))$ is never defective.*

Proof. First assume $s-1=k$. In this case, by Theorem 5.4 (iii), the *sth* higher secant variety of $Seg(\mathbb{P}^k \times X)$ is not defective and we have:

$$\dim\sigma_s(Seg(\mathbb{P}^k \times X)) = \min\{s(n+s)-1,\ s(r+1)-1\}.$$

Now, since $r-n=s-1$, we have

$$s(n+s)-1 = s(r+1)-1,$$

moreover

$$s(n+s)-1 = (k+1)(r+1)-1 = N,$$
$$s(r+1)-1 = s(k+n+1)-1,$$

and so, for $s=k+1$, we have that

$$\dim\sigma_s(Seg(\mathbb{P}^k \times X)) = N = s(k+n+1)-1 = s(\dim Seg(\mathbb{P}^k \times X)+1)-1.$$

Now assume that $s \neq k+1$. In this case, by Remark 1.2, we get:

- for $s > k+1$, $\dim\sigma_s(Seg(\mathbb{P}^k \times X)) = N$;
- for $s < k+1$, $\dim\sigma_s(Seg(\mathbb{P}^k \times X)) = s(k+n+1)-1$,

and the conclusion immediately follows. □

Example 5.6. Let Y be the Segre-Veronese embedding of $\mathbb{P}^{\binom{n+1}{2}} \times \mathbb{P}^n$ via divisors of bi-degree $(1,2)$. Then $\sigma_s Y$ is never defective. In fact, let $X \subset \mathbb{P}^{\binom{n+2}{2}-1}$ be the 2−uple Veronese embedding of $\mathbb{P}^n$. Since

$$Y = Seg(\mathbb{P}^{\binom{n+1}{2}} \times X).$$

and since $\binom{n+1}{2} = \binom{n+2}{2} - 1 - n$, then from Corollary 5.5 we get the conclusion.

Example 5.7. Let Y be the Segre-Veronese embedding of $\mathbb{P}^k \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ via divisors of multi-degree $(1, d_1, \ldots, d_t)$. If $k = \Pi_{i=1}^t \binom{n_i+d_i}{d_i} - \sum_{i=1}^t n_i - 1$, Corollary 5.5 implies that $\sigma_s(Y)$ is never defective.

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