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# ACYCLIC EDGE-COLOURING OF PLANAR GRAPHS\*

MANU BASAVARAJU<sup>†</sup>, L. SUNIL CHANDRAN<sup>‡</sup>, NATHANN COHEN<sup>§</sup>, FRÉDÉRIC HAVET<sup>§</sup>, AND TOBIAS MÜLLER<sup>¶</sup>

**Abstract.** A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph  $G$ , denoted  $\chi'_a(G)$ , is the minimum  $k$  such that  $G$  admits an *acyclic edge-colouring* with  $k$  colours. We conjecture that if  $G$  is planar and  $\Delta(G)$  is large enough then  $\chi'_a(G) = \Delta(G)$ . We settle this conjecture for planar graphs with girth at least 5. We also show that  $\chi'_a(G) \leq \Delta(G) + 12$  for all planar  $G$ , which improves a previous result by Fiedorowicz et al. [12].

**Key words.** spanning galaxy; even strong subdigraph; directed star arboricity

**1. Introduction.** A proper edge-colouring with the property that every cycle contains edges of at least three distinct colours is called an *acyclic edge-colouring*. The *acyclic chromatic index* of a graph  $G$ , denoted  $\chi'_a(G)$ , is the minimum  $k$  such that  $G$  admits an *acyclic edge-colouring* with  $k$  colours. Fiamčík [9] and later Alon, Sudakov and Zaks [2] conjecture that  $\Delta(G) + 2$  colours are enough.

CONJECTURE 1.1 (Fiamčík [9]–Alon, Sudakov and Zaks [2]). *For every graph  $G$ ,  $\chi'_a(G) \leq \Delta(G) + 2$ .*

This conjecture would be tight as there are cases where more than  $\Delta + 1$  colours are needed. Consider for example a graph  $G$  on  $2n$  vertices with at least  $2n^2 - 2n + 2$  edges. The union of two perfect matchings is a cycle factor and thus contains a cycle. Thus, in an acyclic edge-colouring, at most one colour class contains  $n$  edges. Hence there are at least  $1 + \left\lceil \frac{2n^2 - 3n + 2}{n - 1} \right\rceil = 2n + 1$  colours. So  $\chi'_a(G) \geq \Delta(G) + 2$ .

Clearly, every graph with maximum degree at most 2 has acyclic chromatic index at most 3. If  $\Delta(G) \leq 3$  then its line-graph  $L(G)$  has maximum degree at most 4. Thus by Burnstein's results [7]  $\chi_a(L(G)) \leq 5$  and so  $\chi'_a(G) \leq 5$ . So Conjecture 1.1 holds for  $\Delta(G) \leq 3$ . In 1980, Fiamčík [10] conjectured that  $K_4$  is the only cubic graph requiring five colours in an acyclic edge-colouring (and actually gave an uncorrect proof of it). More generally, Alon, Sudakov and Zaks [2] conjectured that if  $G$  is a  $\Delta$ -regular graph then  $\chi'_a(G) = \Delta + 1$  unless  $G = K_{2n}$ .

However as noted by Fiamčík [11], these two conjectures are false as  $\chi'_a(K_{3,3}) = 5$ . In addition, Basavaraju, Chandran and Kummini [5] showed that all  $d$ -regular graphs with  $2n$  vertices and  $d > n$ , require at least  $d + 2$  colours to be acyclically edge-coloured and for every odd  $n$ ,  $\chi'_a(K_{n,n}) = n + 2$ . They also showed that for every  $d, n$  such that  $d \geq 5$ ,  $n \geq 2d + 3$  and  $dn$  even, there exist  $d$ -regular graphs which require at least  $d + 2$ -colours to be acyclically edge-coloured.

Alon, Sudakov and Zaks [2] showed that Conjecture 1.1 is true for almost all regular graphs. This was later improved by Nešetřil and Wormald [19] who proved that the acyclic edge-chromatic number of a random  $\Delta$ -regular graph is asymptotically almost surely equal to  $\Delta + 1$ . Alon, McDiarmid and Reed [1] showed an upper bound of  $64\Delta(G)$  for  $\chi'_a(G)$  which was later improved to  $16\Delta(G)$  by Molloy and Reed [16]. For graphs with large girth, better upper bounds are known. Muthu et al [17] showed that, if  $G$  has girth at least 9, then  $\chi'_a(G) \leq 6\Delta(G)$ , and, if it has girth at least 220, then  $\chi'_a(G) \leq 4.52\Delta(G)$ . Finally, Alon, Sudakov and Saks also showed that Conjecture 1.1 is true for graphs with girth at least  $C\Delta \log(\Delta)$  for some fixed constant  $C$ .

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Muthu et al [18] proved that  $\chi'_a(G) \leq \Delta(G) + 1$  for outerplanar graphs. Fiedorowicz et al. [12] proved that  $\chi'_a(G) \leq 2\Delta(G) + 29$  if  $G$  is planar and  $\chi'_a(G) \leq \Delta(G) + 6$  if  $G$  is planar and triangle-free. This bound has been improved for planar graphs with larger girth. Recall that the *girth* of a graph is the minimum length of a cycle it contains or  $+\infty$  if it has no cycles. Hou et al. [14] showed that if  $G$  is a planar graph  $G$  then  $\chi'_a(G) \leq \Delta(G) + 2$  if  $G$  has girth at least 5,  $\chi'_a(G) \leq \Delta(G) + 1$  if  $G$  has girth at least 7 and  $\chi'_a(G) \leq \Delta(G)$  if  $G$  has girth at least 16 and  $\Delta(G) \geq 3$ .

Sanders and Zhao [20] showed that planar graphs with maximum degree  $\Delta \geq 7$  have chromatic index  $\Delta$ . A conjecture of Vizing [21] asserts that planar graphs of maximum degree 6 are also 6-edge-colourable. This would be best possible as for any  $\Delta \in \{2, 3, 4, 5\}$ , there are some planar graphs with maximum degree  $\Delta$  with chromatic index  $\Delta + 1$  [21].

We propose a conjecture analogous to the above one of Vizing.

**CONJECTURE 1.2.** There exists  $\Delta_0$  such that every planar graph with maximum degree  $\Delta \geq \Delta_0$  has an acyclic edge-colouring with  $\Delta$  colours.

In this paper, we give some evidences to this conjecture. Firstly, in Section 2, we show that every planar graph  $G$  has an acyclic edge-colouring with  $\Delta(G) + 12$  colours thus improving the  $2\Delta(G) + 29$  bound of Fiedorowicz et al. [12]. In Section 3, we show that Conjecture 1.2 holds for planar graphs of girth at least 5 (with  $\Delta_0 = 19$ ) thus improving the results of Hou et al. [14] and Borowiecki and Fiedorowicz [6]. More generally, we settle Conjecture 1.2 for graphs with maximum average degree less than  $4 - \epsilon$  for any  $\epsilon > 0$ . The *maximum average degree* of  $G$  is  $Mad(G) = \max\{\frac{2|E(H)|}{|V(H)|} \mid H \text{ is a subgraph of } G\}$ . It is well known that a planar graph of girth  $g$  has maximum average degree less than  $2 + \frac{4}{g-2}$ . Conjecture 1.2 holds for outerplanar graphs with  $\Delta_0 = 5$  as shown by Hou et al. [15]. Note that  $\sup\{Mad(G) \mid G \text{ is outerplanar}\} = 4$ .

Our proofs are constructive and yield efficient polynomial time algorithms. We present the proofs in a non-algorithmic way. But it is easy to extract the underlying algorithms from them.

**2. Planar graphs.** In this section we will prove the following result.

**THEOREM 2.1.**  $\chi'_a(G) \leq \Delta(G) + 12$  for all planar graphs  $G$ .

The proof of Theorem 2.1 relies on the following theorem of van den Heuvel and McGuiness [13] which establishes a set of unavoidable configurations in planar graphs.

**LEMMA 2.2** (van den Heuvel and McGuiness [13]). *Let  $G$  be a planar graph with minimum degree at least two. Then there exists a vertex  $v$  in  $G$  with exactly  $d(v) = k$  neighbours  $v_1, v_2, \dots, v_k$  with  $d(v_1) \leq d(v_2) \leq \dots \leq d(v_k)$  such that at least one of the following is true:*

- (A1)  $k = 2$ ,
- (A2)  $k = 3$  and  $d(v_1) \leq 11$ ,
- (A3)  $k = 4$  and  $d(v_1) \leq 7, d(v_2) \leq 11$ ,
- (A4)  $k = 5$  and  $d(v_1) \leq 6, d(v_2) \leq 7, d(v_3) \leq 11$ .

*Sketch of the proof of Theorem 2.1:* Let  $G$  be a minimum counter-example with respect to the number of vertices and edges for the statement in Theorem 2.1. Trivially  $G$  has minimum degree at least 2. Indeed, it has no vertex  $v$  of degree 0 because any acyclic edge-colouring of  $G - v$  is an acyclic edge-colouring of  $G$ , and it has no vertex  $v$  with a unique neighbour  $u$ , since any acyclic edge-colouring of  $G - v$  on at least  $\Delta(G)$  colours may be extended to an acyclic edge-colouring of  $G$  by assigning to  $uv$  a colour not already assigned to an edge incident to  $u$ . From Lemma 2.2, we know that there exists a vertex  $v$  in  $G$  such that it belongs to one of the configurations A1–A4. If there is a configuration  $A_2, A_3$  and  $A_4$  in  $G$ , we show in Subsection 2.2 how to derive an acyclic edge-colouring with  $\Delta(G) + 12$  colours of  $G$  from one of  $G \setminus vv_1$ . Hence, we assume that there is no such configurations. In such case, we select an appropriate edge  $uu'$  and show again how to derive an acyclic edge-colouring of  $G$  with  $\Delta(G) + 12$  colours from one of  $G \setminus uu'$ . This gives a final contradiction. See Subsection 2.3.

In order to show how to extend an acyclic edge-colouring of  $G \setminus e$  for some edge  $e$  into an acyclic edge-colouring of  $G$ , we first establish some preliminaries.

## 2.1. Preliminaries.

*Partial edge-colouring:* Let  $H$  be a subgraph of  $G$ . Then an edge-colouring  $c'$  of  $H$  is also a *partial edge-colouring* of  $G$ . Note that  $H$  can be  $G$  itself. Thus an edge-colouring  $c$  of  $G$  itself can be considered a partial edge-colouring. A partial edge-colouring  $c$  of  $G$  is said to be a *proper partial edge-colouring* if  $c$  is proper. A proper partial edge-colouring  $c$  is called *acyclic* if there are no bichromatic cycles in the graph. Note that with respect to a partial edge-colouring  $c$ ,  $c(e)$  may not be defined for an edge  $e$ . So, whenever we use  $c(e)$ , we are considering an edge  $e$  for which  $c(e)$  is defined, though we may not always explicitly mention it.

Let  $c$  be a partial edge-colouring of  $G$ . We denote the set of colours in  $c$  by  $C = \{1, 2, \dots, k\}$ . For any vertex  $u \in V(G)$ , we define  $F_u(c) = \{c(uz) \mid z \in N_G(u)\}$ , with  $N_G(u)$  denotes the set of vertices adjacent to  $u$ . For an edge  $ab \in E$ , we define  $S_{ab}(c) = F_b(c) - \{c(ab)\}$ . Note that  $S_{ab}(c)$  need not be the same as  $S_{ba}(c)$ . We will abbreviate the notation to  $F_u$  and  $S_{ab}$  when the edge-colouring  $c$  is understood from the context.

The following definitions arise out of our attempt to understand what may prevent us from extending a partial edge-colouring of  $G \setminus e$  to  $G$ .

*Maximal bichromatic Path:* An  $(\alpha, \beta)$ -maximal bichromatic path with respect to a partial edge-colouring  $c$  of  $G$  is a maximal path consisting of edges that are coloured using the colours  $\alpha$  and  $\beta$  alternately. An  $(\alpha, \beta, a, b)$ -maximal bichromatic path is an  $(\alpha, \beta)$ -maximal bichromatic path which starts at the vertex  $a$  with an edge coloured  $\alpha$  and ends at  $b$ . We emphasize that the edge of the  $(\alpha, \beta, a, b)$ -maximal bichromatic path incident on vertex  $a$  is coloured  $\alpha$  and the edge incident on vertex  $b$  can be coloured either  $\alpha$  or  $\beta$ . Thus the notations  $(\alpha, \beta, a, b)$  and  $(\alpha, \beta, b, a)$  have different meanings. Also note that any maximal bichromatic path will have at least two edges. The following fact is obvious from the definition of proper edge-colouring.

**FACT 2.3.** *Given a pair of colours  $\alpha$  and  $\beta$  of a proper edge-colouring  $c$  of  $G$ , there is at most one maximal  $(\alpha, \beta)$ -bichromatic path containing a particular vertex  $v$ , with respect to  $c$ .*

A colour  $\alpha \neq c(e)$  is a *candidate* for an edge  $e$  in  $G$  with respect to a partial edge-colouring  $c$  of  $G$  if none of the adjacent edges of  $e$  is coloured  $\alpha$ . A candidate colour  $\alpha$  is *valid* for an edge  $e$  if assigning the colour  $\alpha$  to  $e$  does not result in any bichromatic cycle in  $G$ .

Let  $e = ab$  be an edge in  $G$ . Note that any colour  $\beta \notin F_a \cup F_b$  is a candidate colour for the edge  $ab$  in  $G$  with respect to the partial edge-colouring  $c$  of  $G$ . A sufficient condition for a candidate colour being valid is captured in the lemma below.

**LEMMA 2.4** (Basavaraju and Chandran [4]). *A candidate colour for an edge  $e = ab$  is valid if  $(F_a(c) \cap F_b(c)) \setminus \{c(ab)\} = S_{ab}(c) \cap S_{ba}(c) = \emptyset$ .*

Now even if  $S_{ab}(c) \cap S_{ba}(c) \neq \emptyset$ , a candidate colour  $\beta$  may be valid. But if  $\beta$  is not valid, then what may be the reason? It is clear that colour  $\beta$  is not *valid* if and only if there exists  $\alpha \neq \beta$  such that a  $(\alpha, \beta)$ -bichromatic cycle gets formed if we assign colour  $\beta$  to the edge  $e$ . In other words, if and only if, with respect to edge-colouring  $c$  of  $G$  there existed an  $(\alpha, \beta, a, b)$ -maximal bichromatic path with  $\alpha$  being the colour given to the first and last edge of this path. Such paths play an important role in our proofs. We call them *critical paths*. It is formally defined below.

*Critical Path:* Let  $ab \in E$  and  $c$  be a partial edge-colouring of  $G$ . Then an  $(\alpha, \beta, a, b)$ -maximal bichromatic path which starts out from the vertex  $a$  via an edge coloured  $\alpha$  and ends at the vertex  $b$  via an edge coloured  $\alpha$  is called an  $(\alpha, \beta, a, b)$ -critical path. Note that any critical path will be of odd length. Moreover the smallest length possible is three.

Let  $a \in N_{G \setminus v_1}(x)$  and let  $c(x, a) = \alpha$ . Let  $\beta \in S_{xa}$ . colour  $\beta$  is said to be *actively present* in a set  $S_{xa}$ , if there exists a  $(\alpha, \beta, xy)$  critical path.

A natural strategy to extend a acyclic partial edge-colouring  $c$  of  $G$  would be to try to assign one of the candidate colours to an uncoloured edge  $e$ . The condition that a candidate colour is not valid for the edge  $e$  is captured in the following fact.

**FACT 2.5.** *Let  $c$  be a partial edge-colouring of  $G$ . A candidate colour  $\beta$  is not valid for the edge*

$e = ab$  if and only if for some colour  $\alpha \in S_{ab} \cap S_{ba}$ , there is an  $(\alpha, \beta, a, b)$ -critical path in  $G$  with respect to  $c$ .

*Colour exchange.*: Let  $c$  be a partial edge-colouring of  $G$ . Let  $u, v, w \in V(G)$  and  $uv, uw \in E(G)$ . We define *colour exchange* with respect to the edge  $uv$  and  $uw$ , as the modification of the current partial edge-colouring  $c$  by exchanging the colours of the edges  $uv$  and  $uw$  to get a partial edge-colouring  $c'$ , i.e.,  $c'(uv) = c(uw)$ ,  $c'(uw) = c(uv)$  and  $c'(e) = c(e)$  for all other edges  $e$  in  $G$ . The colour exchange with respect to the edges  $uv$  and  $uw$  is said to be *proper* (resp. *acyclic*) if the edge-colouring obtained after the exchange is proper (resp. acyclic). The following fact is obvious.

**FACT 2.6.** *Let  $c'$  be the partial edge-colouring obtained from an acyclic partial edge-colouring  $c$  by the colour exchange with respect to the edges  $uv$  and  $uw$ . Then  $c'$  is proper if and only if  $c(uv) \notin S_{uv}$  and  $c(uw) \notin S_{uv}$ .*

The colour exchange is useful in breaking some critical paths as is clear from the following lemma.

**LEMMA 2.7** (Basavaraju and Chandran [4, 3]). *Let  $u, v, w, a$  and  $b$  be vertices of  $G$  such that  $uv, uw$  and  $ab$  are edges. Also let  $\alpha$  and  $\beta$  be two colours such that  $\{\alpha, \beta\} \cap \{c(uv), c(uw)\} \neq \emptyset$  and  $\{v, w\} \cap \{a, b\} = \emptyset$ . Suppose there exists a  $(\alpha, \beta, a, b)$ -critical path that contains vertex  $u$ , with respect to an acyclic partial edge-colouring  $c$  of  $G$ . Let  $c'$  be the partial edge-colouring obtained from  $c$  by the colour exchange with respect to the edges  $uv$  and  $uw$ . If  $c'$  is proper, then there is no  $(\alpha, \beta, a, b)$ -critical path in  $G$  with respect to  $c'$ .*

*Multisets and Multiset Operations.*: Recall that a *multiset* is a generalized set where a member can appear multiple times. If an element  $x$  appears  $t$  times in the multiset  $S$ , then we say that the *multiplicity* of  $x$  in  $S$  is  $t$ . In notation  $\text{mult}_S(x) = t$ . The cardinality of a finite multiset  $S$ , denoted by  $\|S\|$ , is defined as  $\|S\| = \sum_{x \in S} \text{mult}_S(x)$ . Let  $S_1$  and  $S_2$  be two multisets. The reader may note that there are various possible ways to define union of  $S_1$  and  $S_2$ . For the purpose of this paper we define one such union notion- which we call as the *join* of  $S_1$  and  $S_2$ , denoted as  $S_1 \uplus S_2$ . The multiset  $S_1 \uplus S_2$  have all the members of  $S_1$  as well as  $S_2$ . For a member  $x \in S_1 \uplus S_2$ ,  $\text{mult}_{S_1 \uplus S_2}(x) = \text{mult}_{S_1}(x) + \text{mult}_{S_2}(x)$ . Clearly  $\|S_1 \uplus S_2\| = \|S_1\| + \|S_2\|$ .

**2.2. There exists a Configuration A2, A3 or A4.** We now can resume the proof of Theorem 2.1. Suppose by way of contradiction that there exists a Configuration  $A_2, A_3$  or  $A_4$  in  $G$ . Let  $v, v_1, v_2$  and  $v_3$  be the vertices as described in Lemma 2.2.

In all the propositions of this subsection, we start with an acyclic edge-colouring  $c'$  of  $G \setminus vv_1$ . So the abbreviations  $F_u$  and  $S_{ab}$  stand for  $F_u(c')$  and  $S_{ab}(c')$  respectively.

**PROPOSITION 2.8.** *For any acyclic edge-colouring  $c'$  of  $G \setminus vv_1$ ,  $|F_v \cap F_{v_1}| \geq 2$ .*

*Proof.* Suppose by way of contradiction that there is an acyclic edge-colouring  $c'$  of  $G \setminus vv_1$  with a set  $C$  of  $\Delta + 12$  colours such that  $|F_v \cap F_{v_1}| \leq 1$ .

Assume first that  $|F_v \cap F_{v_1}| = 0$ . The reader can verify from close examination of Configurations A2, A3 and A4 that  $|F_v \cup F_{v_1}|$  will be maximum for Configuration A2 and therefore  $|F_v \cup F_{v_1}| = |F_v| + |F_{v_1}| \leq 2 + 10 = 12$ . Thus there are  $\Delta$  candidate colours for the edge  $vv_1$  and by Lemma 2.4 all the candidate colours are valid, a contradiction to the assumption that  $G$  is a counter-example.

Assume now that  $|F_v \cap F_{v_1}| = 1$ . It is easy to see that  $|F_v \cup F_{v_1}| = |F_v| + |F_{v_1}| - |F_v \cap F_{v_1}| \leq 11$  and hence there are at least  $\Delta + 1$  candidate colours for the edge  $vv_1$ . Let  $F_v \cap F_{v_1} = \{\alpha\}$  and let  $u \in N(v)$  be a vertex such that  $c'(vu) = \alpha$ . Now if none of the  $\Delta + 1$  candidate colours is valid for the edge  $vv_1$ , then by Fact 2.5, for each  $\gamma \in C \setminus (F_v \cup F_{v_1})$ , there exists an  $(\alpha, \gamma, v, v_1)$ -critical path. Since  $c'(vu) = \alpha$ , we have all the critical paths passing through the vertex  $u$  and hence  $S_{vu} \subseteq C \setminus (F_v \cup F_{v_1})$ . This implies that  $|S_{vu}| \geq |C \setminus (F_v \cup F_{v_1})| \geq (\Delta + 12) - 11 = \Delta + 1$ , a contradiction since  $|S_{vu}| \leq \Delta - 1$ . Thus we have a valid colour for the edge  $vv_1$ , a contradiction to the assumption that  $G$  is a counter-example.  $\square$

Let  $S_v$  be the multiset defined by  $S_v = S_{vv_2} \uplus S_{vv_3} \uplus \dots \uplus S_{vv_k}$ .

**PROPOSITION 2.9.** *For any acyclic edge-colouring  $c'$  of  $G \setminus vv_1$ ,  $|F_v \cap F_{v_1}| \neq 2$ .*

*Proof.* Suppose not. Let  $F_v \cap F_{v_1} = \{\alpha_1, \alpha_2\}$  and let  $v', v'' \in N_{G \setminus vv_1}(v)$  and  $u', u'' \in N_{G \setminus vv_1}(v_1)$  be such that  $c'(vv') = c'(v_1u') = \alpha_1$  and  $c'(vv'') = c'(v_1u'') = \alpha_2$ . It is easy to see that  $|F_v \cup F_{v_1}| \leq 10$ .

Thus there are at least  $\Delta + 2$  candidate colours for the edge  $vv_1$ . If any of the candidate colours is valid for the edge  $vv_1$ , we are done. Thus none of the candidate colours is valid for the edge  $vv_1$ . This implies that there exists a  $(\alpha_1, \theta, v, v_1)$ - or  $(\alpha_2, \theta, v, v_1)$ -critical path for each candidate colour  $\theta$ .

CLAIM 1. *The multiset  $S_v$  contains at least  $|F_{v_1}| - 1$  colours from  $F_{v_1}$ .*

*Proof.* Suppose not. Then there are at least two colours in  $F_{v_1}$  which are not in  $S_v$ . Let  $\nu$  and  $\mu$  be any two such colours. Now assign colours  $\nu$  and  $\mu$  to the edges  $vv'$  and  $vv''$  respectively to get an edge-colouring  $c''$ . Now since  $\nu, \mu \notin S_v$ , we have  $\nu \notin S_{vv'}$  and  $\mu \notin S_{vv''}$ . Moreover  $\mu, \nu \notin F_v(c'') \setminus \{\alpha_1, \alpha_2\}$ . Thus the edge-colouring  $c''$  is proper. Now we claim that the edge-colouring  $c''$  is acyclic also. Suppose not. Then there has to be a bichromatic cycle containing at least one of the colours  $\nu$  and  $\mu$ . Clearly this cannot be a  $(\nu, \mu)$ -bichromatic cycle since  $\mu \notin S_{vv'}$ . Therefore it has to be a  $(\nu, \lambda)$ - or  $(\mu, \lambda)$ -bichromatic cycle where  $\lambda \in F_v(c'') \setminus \{\nu, \mu\}$ . Let  $u$  be a vertex such that  $c''(vu) = \lambda$ . This means that there was already a  $(\lambda, \nu, v, v')$ - or  $(\lambda, \mu, v, v'')$ -critical path with respect to the edge-colouring  $c'$ . This implies that  $\nu \in S_{vu}$  or  $\mu \in S_{vu}$ , implying that  $\nu \in S_v$  or  $\mu \in S_v$ , a contradiction. Thus the edge-colouring  $c''$  is acyclic. Let  $u_1, u_2 \in N_{G \setminus vv_1}(v_1)$  be such that  $c''(v_1u_1) = \nu$  and  $c''(v_1u_2) = \mu$ .

Note that  $|F_v \cup F_{v_1}| \leq 10$  (The maximum value of  $|F_v \cup F_{v_1}|$  is attained when the graph has Configuration A2). Therefore there are at least  $\Delta + 2$  candidate colours for the edge  $vv_1$ . If any of the candidate colours are valid for the edge  $vv_1$ , then we are done as this is a contradiction to the assumption that  $G$  is a counter-example. Thus none of the candidate colours is valid for the edge  $vv_1$  and therefore there exist either a  $(\nu, \theta, v, v_1)$ -critical or a  $(\mu, \theta, v, v_1)$ -critical path for each candidate colour  $\theta$ . Let  $C_\nu$  and  $C_\mu$  respectively be the set of candidate colours which are forming critical paths with colours  $\nu$  and  $\mu$ . Then clearly  $C_\nu \subseteq S_{v_1u_1}$  and  $C_\mu \subseteq S_{v_1u_2}$  since  $c''(v_1u_1) = \nu$  and  $c''(v_1u_2) = \mu$ . Now we exchange the colours of the edges  $vv'$  and  $vv''$  to get a modified edge-colouring  $c$ . Note that  $c$  is proper since  $\mu \notin S_{vv'}$  and  $\nu \notin S_{vv''}$ . By Lemma 2.7, all  $(\nu, \beta, v, v_1)$ -critical paths where  $\beta \in C_\nu$  and all  $(\mu, \gamma, v, v_1)$ -critical paths where  $\gamma \in C_\mu$  are broken. Now if none of the colours in  $C_\nu$  are valid for edge  $vv_1$ , then it means that for each  $\beta \in C_\nu$ , there exists a  $(\mu, \beta, v, v_1)$ -critical path with respect to the edge-colouring  $c$ , implying that  $C_\nu \subseteq S_{v_1u_2}$ . Since the recolouring involved no candidate colours, we still have  $C_\mu \subseteq S_{v_1u_2}$ . Thus we have  $(C_\nu \cup C_\mu) \subseteq S_{v_1u_2}$ . But  $|C_\nu \cup C_\mu| \geq \Delta + 2$  which implies that  $|S_{v_1u_2}| \geq \Delta + 2$ , a contradiction since  $|S_{v_1u_2}| \leq \Delta - 1$ .  $\square$

CLAIM 2. *There exists at least two colours  $\beta_1$  and  $\beta_2$  in  $C \setminus F_{v_1}$  with multiplicity at most one in  $S_v$ .*

*Proof.* In view of Claim 1 we have  $\sum_{x \in C \setminus F_{v_1}} \text{mult}_{S_v}(x) = \|S_v\| - (|F_v| - 1)$ . Thus if  $\|S_v\| - (|F_{v_1}| - 1) \leq 2(|C \setminus F_{v_1}|) - 3$ , then there exist at least two colours  $\beta_1$  and  $\beta_2$  in  $C \setminus F_{v_1}$  with multiplicity at most one in  $S_v$ . Thus it is enough to prove  $\|S_v\| \leq 2|C| - |F_{v_1}| - 4 \leq 2\Delta + 24 - |F_{v_1}| - 4 = 2\Delta + 20 - |F_{v_1}|$ . Now we can easily verify that  $\|S_v\| + |F_{v_1}| \leq 2\Delta + 20$  for Configurations A2, A3 and A4 as follows:

- For A2,  $\|S_v\| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + |F_{v_1}| = (\Delta - 1) + (\Delta - 1) + 10 = 2\Delta + 8$ .
- For A3,  $\|S_v\| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + |F_{v_1}| = 10 + (\Delta - 1) + (\Delta - 1) + 6 = 2\Delta + 14$ .
- For A4,  $\|S_v\| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + (d(v_5) - 1) + |F_{v_1}| = 6 + 10 + (\Delta - 1) + (\Delta - 1) + 5 = 2\Delta + 19$ .

$\square$

The colours  $\beta_1$  and  $\beta_2$  of Claim 2 are crucial to the proof. Now we make another claim regarding  $\beta_1$  and  $\beta_2$ :

CLAIM 3.  *$\beta_1$  and  $\beta_2 \in F_v$ .*

*Proof.* Without loss of generality, let  $\beta_1 \notin F_v$ . Then recalling that  $\beta_1 \notin F_{v_1}$ ,  $\beta_1$  is a candidate for the edge  $vv_1$ . If it is not valid, then there exists either an  $(\alpha_1, \beta_1, vv_1)$ - or  $(\alpha_2, \beta_1, v, v_1)$ -critical path with respect to  $c'$ . Since the multiplicity of  $\beta_1$  in  $S_v$  is at most one, we have the colour  $\beta_1$  in exactly one of  $S_{vv'}$  or  $S_{vv''}$ . Without loss of generality let  $\beta_1 \in S_{vv''}$ . Hence there exists an  $(\alpha_2, \beta_1, v, v_1)$ -critical path with respect to  $c'$ .

Now recolour the edge  $vv'$  with colour  $\beta_1$  to get an edge-colouring  $c$ . Then  $c$  is proper since  $\beta_1 \notin F_v$  and  $\beta_1 \notin S_{vv'}$ . We shall prove that  $c$  is acyclic. Suppose, by way of contradiction, that there is

a bichromatic cycle with respect to  $c$ . Then it has to be a  $(\beta_1, \gamma)$ -bichromatic cycle for some  $\gamma \in F_v(c) \setminus c(vv')$ . Let  $a \in N_{G \setminus vv_1}(v)$  be such that  $c(va) = \gamma$ . Then the  $(\beta_1, \gamma)$ -bichromatic cycle should contain the edge  $va$  and therefore  $\gamma \in S_{va}(c)$ . But we know that  $v''$  is the only vertex in  $N_{G \setminus vv_1}(v)$  such that  $\beta_1 \in S_{vv''}$ . Therefore  $a = v''$ . This implies that  $\gamma = \alpha_2$  and there existed an  $(\alpha_2, \beta_1, v, v')$ -critical path with respect to the edge-colouring  $c'$ . This is a contradiction to Fact 2.3 since there already existed an  $(\alpha_2, \beta_1, v, v_1)$ -critical path with respect to the edge-colouring  $c'$ .

Thus the edge-colouring  $c$  is acyclic and  $|F_v(c) \cap F_{v_1}(c)| = 1$ , a contradiction to Proposition 2.8.  $\square$

Note that  $\{\beta_1, \beta_2\} \cap \{\alpha_1, \alpha_2\} = \emptyset$  since  $\beta_1, \beta_2 \notin F_{v_1}$ . In view of Claim 3, we have  $\{\alpha_1, \alpha_2, \beta_1, \beta_2\} \subseteq F_v$  and thus  $|F_v| \geq 4$ , which implies that  $d(v) \geq 5$ . Thus the vertex  $v$  belongs to Configuration A4. Therefore  $d(v) = 5$  and  $F_v = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ . There are at least  $\Delta + 12 - (5 + 4 - 2) = \Delta + 5$  candidate colours for the edge  $vv_1$ . Also recall that  $d(v_2) \leq 7$ ,  $c'(vv') = c'(v_1u') = \alpha_1$  and  $c'(vv'') = c'(v_1u'') = \alpha_2$ .

CLAIM 4.  $v_2 \notin \{v', v''\}$ .

*Proof.* Suppose not. Then, without loss of generality,  $v_2 = v'$  and  $c'(vv_2) = \alpha_1$ . Now if none of the  $\Delta + 5$  candidate colours is valid for the edge  $vv_1$ , then they all are in critical paths that contain either the edge  $vv'$  or the edge  $vv''$ . Now  $|S_{vv'}| + |S_{vv''}| \leq 6 + \Delta - 1 = \Delta + 5$ . Since each of the  $\Delta + 5$  candidate colours has to be present in either in  $S_{vv'}$  or  $S_{vv''}$ , we infer that  $S_{vv'} \cup S_{vv''}$  is exactly the set of candidate colours, i.e.,  $|S_{vv'}| + |S_{vv''}| = \Delta + 5$ . This requires that  $|S_{vv'}| = 6$ ,  $|S_{vv''}| = \Delta - 1$  and  $S_{vv'} \cap S_{vv''} = \emptyset$ . Since for each  $\gamma \in S_{vv''}$ , we have  $(\alpha_2, \gamma, v, v_1)$ -critical path containing  $u''$ , we can infer that  $S_{vv''} \subseteq S_{v_1u''}$  (Recall that  $c'(v_1u'') = \alpha_2$ ). But since  $|S_{v_1u''}| \leq \Delta - 1$ , we have  $S_{vv''} = S_{v_1u''}$ . Thus  $S_{v_1u''} \cap S_{vv'} = S_{vv''} \cap S_{vv'} = \emptyset$ .

Now we exchange the colours of the edges  $vv'$  and  $vv''$  to get an edge-colouring  $c$ . Hence  $c(vv') = \alpha_2$  and  $c(vv'') = \alpha_1$ . The edge-colouring  $c$  is proper since  $\alpha_2 \notin S_{vv'}$  and  $\alpha_1 \notin S_{vv''}$  (Recall that  $S_{vv'}$  and  $S_{vv''}$  contain only candidate colours). We shall prove that  $c$  is also acyclic: A bichromatic cycle with respect to  $c$  has to be an  $(\alpha_1, \eta)$ - or  $(\alpha_2, \eta)$ -bichromatic cycle for some  $\eta \in F_v$ . Clearly it cannot be an  $(\alpha_1, \alpha_2)$ -bichromatic cycle since  $\alpha_1 \notin S_{vv'}(c)$  and therefore  $\eta \in \{\beta_1, \beta_2\}$  (Recall that  $F_v = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ ). This implies that either  $\beta_1$  or  $\beta_2$  belongs to  $S_{vv'} \cup S_{vv''}$ . But we know that  $S_{vv'} \cup S_{vv''}$  is exactly the set of candidate colours for the edge  $vv_1$ , a contradiction since  $\beta_1, \beta_2 \in F_v$  cannot be candidate colours for the edge  $vv_1$ .

Therefore the edge-colouring  $c$  is acyclic. By Lemma 2.7, all the existing critical paths are broken. Now consider a colour  $\gamma \in S_{vv'}$ . If it is still not valid then there has to be a  $(\alpha_2, \gamma, v, v_1)$ -critical path since  $c(vv') = \alpha_2$  and  $\gamma \notin S_{vv''}(c)$ . This implies that  $\gamma \in S_{v_1u''}(c)$ , a contradiction since  $S_{v_1u''}(c) \cap S_{vv'}(c) = \emptyset$ . Thus we have a valid colour for the edge  $vv_1$ , a contradiction to the assumption that  $G$  is a counterexample.  $\square$

From Claim 4, we infer that  $c'(vv_2) \notin F_v \cap F_{v_1}$  since  $F_v \cap F_{v_1} = \{c'(vv'), c'(vv'')\} = \{\alpha_1, \alpha_2\}$ . Therefore we have  $c'(vv_2) \in \{\beta_1, \beta_2\}$  since  $F_v = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ . Without loss of generality let  $c'(vv_2) = \beta_1$ . We know that the colour  $\beta_2$  can be in at most one of  $S_{vv'}$  and  $S_{vv''}$  by Claim 2. Now let  $v'$  be such that  $\beta_2 \notin S_{vv'}$ . Note that  $C \setminus (S_{vv'} \cup F_v \cup F_{v_1}) \neq \emptyset$  since  $|S_{vv'} \cup F_v \cup F_{v_1}| \leq \Delta - 1 + 4 + 5 - 2 = \Delta + 6$ . Assign a colour  $\theta \in C \setminus (S_{vv'} \cup F_v \cup F_{v_1})$  to the edge  $vv'$  to get an edge-colouring  $c''$ . Now  $|F_v(c'') \cap F_{v_1}(c'')| = 1$ . Thus in view of Proposition 2.8, the edge-coloring  $c''$  is not acyclic. Hence there is a bichromatic cycle with respect to  $c''$ . This bichromatic cycle should involve one of the colours  $\alpha_2, \beta_1, \beta_2$  along with  $\theta$ . Since the bichromatic cycle contains a colour from  $S_{vv'}$  and  $\beta_2 \notin S_{vv'}$ , it cannot be a  $(\theta, \beta_2)$ -bichromatic cycle. Now with respect to the edge-colouring  $c'$ , colour  $\theta$  was not valid for the edge  $vv_1$  implying that there existed a  $(\alpha_1, \theta, v, v_1)$ - or  $(\alpha_2, \theta, v, v_1)$ -critical path. But  $(\alpha_1, \theta, v, v_1)$ -critical path was not possible since  $\theta \notin S_{vv'}$  by the choice of  $\theta$ . Thus there existed an  $(\alpha_2, \theta, v, v_1)$ -critical path with respect to  $c'$ . Thus by Fact 2.3, there cannot be an  $(\alpha_2, \theta, v, v')$ -critical path with respect to  $c'$  and hence there cannot be an  $(\alpha_2, \theta)$ -bichromatic cycle in  $c''$  formed due to the recolouring. Thus if there is a bichromatic cycle formed, then it has to be a  $(\beta_1, \theta)$ -bichromatic cycle, which implies that  $\beta_1 \in S_{vv'}$ .

Now taking into account the fact that  $\beta_1$  is in  $S_{vv'}$  as well as  $F_v$ , we get  $|S_{vv'} \cup F_v \cup F_{v_1}| \leq \Delta - 1 + 4 + 5 - 2 - 1 = \Delta + 5$  and therefore  $|S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2}| \leq \Delta + 5 + 6 = \Delta + 11$ . Thus

$C \setminus (S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2}) \neq \emptyset$ . Now recolour the edge  $vv'$  using a colour  $\gamma \in C \setminus (S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2})$  to get an edge-colouring  $c$ . Clearly this edge-colouring is proper. It is also acyclic since if a bichromatic cycle gets formed it has to be a  $(\beta_1, \gamma)$  bichromatic cycle (Note that the  $(\alpha_2, \gamma)$  and  $(\beta_2, \gamma)$  bichromatic cycles are argued out as before). But  $\gamma \notin S_{vv_2}$ , a contradiction. Thus the edge-colouring  $c$  is acyclic.

But  $|F_v(c) \cap F_{v_1}(c)| = 1$ , a contradiction to Proposition 2.8. This completes the proof of Proposition 2.9.  $\square$

**PROPOSITION 2.10.** *For any acyclic edge-colouring  $c'$  of  $G \setminus vv_1$ ,  $|F_v \cap F_{v_1}| \neq 3$ .*

*Proof.* Suppose not. Let  $c'$  be an acyclic edge-colouring of  $G \setminus vv_1$  such that  $|F_v \cap F_{v_1}| = 3$ . Then  $|F_v| \geq 3$  and therefore  $d(v) \geq 4$ . Thus  $v$  belongs to either configuration A3 or A4. Let  $S'_v$  be the multiset defined by  $S'_v = S_v \setminus (F_{v_1} \cup F_v)$ . Let  $v', v'', v''' \in N_{G \setminus vv_1}(v)$  be such that  $\{c(vv'), c(vv''), c(vv''')\} = F_v \cap F_{v_1}$ . Also let  $c(vv') = \alpha_1$ ,  $c(vv'') = \alpha_2$  and  $c(vv''') = \alpha_3$ .

**CLAIM 5.**  $\|S'_v\| \leq 2\Delta + 11$ .

*Proof.* When  $d(v) = 4$ , it is clear that  $\|S'_v\| \leq (d(v_2)-1) + (d(v_3)-1) + (d(v_4)-1) \leq 10 + \Delta - 1 + \Delta - 1 = 2\Delta + 8$ . On the other hand when  $d(v) = 5$ , try to recolour one of the edges  $vv'$ ,  $vv''$ ,  $vv'''$  using a colour in  $C \setminus (F_v \cup F_{v_1})$ . There are  $\Delta + 6$  colours in  $C \setminus (F_v \cup F_{v_1})$ . If any of these colours is valid for one of  $vv'$ ,  $vv''$  or  $vv'''$ , then recolouring this edge with this colour, we obtain an acyclic edge-colouring  $c''$  satisfying  $|F_v(c'') \cap F_{v_1}(c'')| = 2$ . This contradicts Proposition 2.9. Hence there has to be a bichromatic cycle formed during each recolouring. Since such a bichromatic cycle has to be a  $(\gamma_1, \gamma_2)$ -bichromatic cycle where  $\gamma_1$  is the colour used in the recolouring and  $\gamma_2 \in F_v \setminus \{\gamma_1\}$ , we infer that  $S_{vv'}$ ,  $S_{vv''}$  and  $S_{vv'''}$  contain at least one colour from  $F_v$ . Thus we have  $\|S'_v\| \leq \|S_v\| - 3 \leq (d(v_2)-1) + (d(v_3)-1) + (d(v_4)-1) + (d(v_5)-1) - 3 \leq 6 + 10 + \Delta - 1 + \Delta - 1 - 3 = 2\Delta + 11$ .  $\square$

**CLAIM 6.** *There exists at least one colour  $\beta \in C \setminus (F_v \cup F_{v_1})$  with multiplicity at most one in  $S'_v$ .*

*Proof.* Since  $v$  belongs to either configuration A3 or configuration A4, we have  $|F_v \cup F_{v_1}| \leq 9 - 3 = 6$ . Thus  $|C \setminus (F_v \cup F_{v_1})| \leq \Delta + 6$ . By Claim 5 we have  $\|S'_v\| \leq 2\Delta + 11$  and from this it is easy to see that there exists at least one colour  $\beta \in C \setminus (F_v \cup F_{v_1})$  with multiplicity at most one in  $S'_v$ .  $\square$

Note that  $\beta \in C \setminus (F_v \cup F_{v_1})$ , where  $\beta$  is the colour from Claim 6 is a candidate colour for the edge  $vv_1$ . If it is not valid then there has to be a  $(\theta, \beta, v, v_1)$ -critical path, where  $\theta \in \{\alpha_1, \alpha_2, \alpha_3\}$ . By Claim 6,  $\beta$  can be present in at most one of  $S_{vv'}$ ,  $S_{vv''}$  and  $S_{vv'''}$ . Without loss of generality let  $\beta \in S_{vv''}$ . Thus there exists an  $(\alpha_2, \beta, v, v_1)$ -critical path with respect to the edge-colouring  $c'$ . Recolour the edge  $vv'$  using the colour  $\beta$  to get an edge-colouring  $c$ . Clearly  $c$  is proper since  $\beta \notin S_{vv'}$  and  $\beta \notin F_v$ . Let us show that it is also acyclic. A bichromatic cycle (with respect to  $c$ ) has to contain the colour  $\beta$  as well as a colour  $\gamma \in F_v(c) \setminus \{\beta\}$ . If  $\gamma = c(vv)$ , then  $\beta \in S_{vw}$ , for the  $(\beta, \gamma)$ -bichromatic cycle to get formed. But  $v''$  is the only vertex in  $N_{G \setminus vv_1}(v)$  such that  $\beta \in S_{vv''}$ . Thus  $w = v''$ ,  $\gamma = \alpha_2$  and the cycle is an  $(\alpha_2, \beta)$ -bichromatic cycle. This means that there existed an  $(\alpha_2, \beta, v, v')$ -critical path with respect to the edge-colouring  $c'$ , a contradiction to Fact 2.3 since there already existed an  $(\alpha_2, \beta, v, v_1)$ -critical path with respect to the edge-colouring  $c'$ . Thus the edge-colouring  $c$  is acyclic.

But  $|F_v(c) \cap F_{v_1}(c)| = 2$ , a contradiction to Proposition 2.9. This completes the proof of Proposition 2.10.  $\square$

**PROPOSITION 2.11.** *For any acyclic edge-colouring  $c'$  of  $G \setminus vv_1$ ,  $|F_v \cap F_{v_1}| \neq 4$ .*

*Proof.* Suppose not. Let  $c'$  be an acyclic edge-colouring of  $G \setminus vv_1$  such that  $|F_v \cap F_{v_1}| = 4$ . Then  $|F_v| \geq 4$  and since  $d(v) \leq 5$ , we have  $d(v) = 5$ . Hence  $v$  belongs to Configuration A4. Let  $S'_v$  be the multiset defined by  $S'_v = S_v \setminus (F_{v_1} \cup F_v)$ . Also let  $c(vv_2) = \alpha_1$ ,  $c(vv_3) = \alpha_2$ ,  $c(vv_4) = \alpha_3$  and  $c(vv_5) = \alpha_4$ .

Now try to recolour an edge incident on  $v$  with a candidate colour from  $C \setminus (F_v \cup F_{v_1})$ . If the obtained edge-colouring  $c''$  is acyclic then  $|F_v(c'') \cap F_{v_1}(c'')| = 3$ , a contradiction to Proposition 2.10. Hence there has to be a bichromatic cycle created due to recolouring with one of the colours from  $F_v$ . This implies that  $F_v \cap S'_v \neq \emptyset$ . Thus we have  $\|S'_v\| \leq \|S_v\| - 1 \leq (d(v_2)-1) + (d(v_3)-1) + (d(v_4)-1) + (d(v_5)-1) \leq 6 + 10 + \Delta - 1 + \Delta - 1 - 1 = 2\Delta + 13$ . Now since there are  $|C \setminus (F_v \cup F_{v_1})| \geq \Delta + 12 - (4 + 5 - 4) = \Delta + 7$  candidate colours and  $\|S'_v\| \leq 2\Delta + 13$ , it is easy to see that there exists at least one candidate colour  $\beta$  with multiplicity at most one in  $S'_v$ .



Note that  $\beta \in C \setminus (F_v \cup F_{v_1})$  is a candidate colour for the edge  $vv_1$ . If it is not valid then there has to be a  $(\theta, \beta, v, v_1)$ -critical path, where  $\theta \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ . We know that  $\beta$  can be present in at most one of  $S_{vv_2}, S_{vv_3}, S_{vv_4}$  and  $S_{vv_5}$ . Without loss of generality let  $\beta \in S_{vv_3}$ . Thus there exists an  $(\alpha_2, \beta, v, v_1)$ -critical path with respect to the edge-colouring  $c'$ . Recolour the edge  $vv_2$  using the colour  $\beta$  to get an edge-colouring  $c$ . Clearly  $c$  is proper since  $\beta \notin S_{vv_2}$  and  $\beta \notin F_v$ . Let us now show that it is acyclic. A bichromatic cycle with respect to  $c$  has to contain the colour  $\beta$  as well as a colour  $\gamma \in F_v(c) \setminus \{\beta\}$ . If  $\gamma = c(vw)$ , then  $\beta \in S_{vw}$ , for the  $(\beta, \gamma)$  bichromatic cycle to get formed. But  $v_3$  is the only vertex in  $N_{G \setminus vv_1}(v)$  such that  $\beta \in S_{vv_3}$ . Thus  $w = v_3$ ,  $\gamma = \alpha_2$  and it has to be a  $(\beta, \alpha_2)$  bichromatic cycle. This means that there existed an  $(\alpha_2, \beta, v, v_2)$ -critical path with respect to the edge-colouring  $c'$ , a contradiction to Fact 2.3 since there already existed an  $(\alpha_2, \beta, v, v_1)$ -critical path with respect to the edge-colouring  $c'$ . Thus the edge-colouring  $c$  is acyclic.

But  $|F_v(c) \cap F_{v_1}(c)| = 3$ , a contradiction to Proposition 2.10.  $\square$

By Lemma 2.2,  $d_{G \setminus vv_1}(v) \leq 4$ . Thus  $|F_v \cap F_{v_1}| \leq |F_v| \leq 4$ . Then Propositions 2.8, 2.9, 2.10 and 2.11 gives a contradiction to the assumption that  $G$  contains a Configuration A2, A3 or A4.

**2.3. There is no Configuration A2, A3 or A4.** In the previous subsection, we showed that  $G$  contains no Configuration A2, A3 or A4. Then by Lemma 2.2, there is a Configuration A1, that is a vertex  $v$  such that  $d(v) = 2$ . Now delete all the degree 2 vertices from  $G$  to get a graph  $H$ . Now since the graph  $H$  is also planar, there exists a vertex  $v'$  in  $H$  such that  $v'$  belongs to one of the configurations A1, A2, A3 or A4, say  $A'$ . The vertex  $v'$  was not already in Configuration  $A'$  in  $G$ . This means that the degree of at least one of the vertices of the configuration  $A'$  i.e.,  $\{v'\} \cup N_H(v')$ , got decreased by the removal of 2-degree vertices. Let  $P = \{x \in \{v'\} \cup N_H(v') : d_H(x) < d_G(x)\}$ . Let  $u$  be the minimum degree vertex in  $P$  in the graph  $H$ . Now it is easy to see that  $d_H(u) \leq 11$  since  $v'$  did not belong to  $A'$  in  $G$ .

Let  $N'(u) = \{x | x \in N_G(u) \text{ and } d_G(x) = 2\}$ . Let  $N''(u) = N_G(u) - N'(u)$ . It is obvious that  $N''(u) = N_H(u)$ .

Since  $u \in P$  and  $d_H(u) \leq 11$ , we have  $|N'(u)| \geq 1$  and  $N''(u) \leq 11$ . In  $G$  let  $u' \in N'(u)$  be a two degree neighbour of  $u$  such that  $N(u') = \{u, u''\}$ . Now by minimality of  $G$ , the graph  $G \setminus uu'$  admits an acyclic edge-colouring  $c'$  using a set  $C$  of  $\Delta + 12$  colours. Let  $F'_u = \{c'(ux) | x \in N'(u)\}$  and  $F''_u = \{c'(ux) | x \in N''(u)\}$ . Now if  $c(u'u'') \notin F_u$  we are done since  $|F_u \cup F_{u'}| \leq \Delta$  and thus there are at least 12 candidate colours which are also valid by Lemma 2.4.

We know that  $|F''_v| \leq 11$ . If  $c'(u'u'') \in F'_v$ , then let  $c = c'$ . Else if  $c'(u'u'') \in F''_v$ , then recolour edge  $u'u''$  using a colour from  $C \setminus (S_{u'u''} \cup F''_v)$  to get an edge-colouring  $c$  (Note that  $|C \setminus (S_{u'u''} \cup F''_v)| \geq \Delta + 12 - (\Delta - 1 + 11) = 2$  and since  $u'$  has degree one in  $G - \{uu'\}$ ,  $c$  is acyclic). Now if  $c(u'u'') \notin F_u$  the proof is already discussed. Thus  $c(u'u'') \in F'_u$ .

Let us now consider the edge-colouring  $c$ . Let  $a \in N'(u)$  be such that  $c(ua) = c(u'u'') = \alpha$ . Now if none of the candidate colours in  $C \setminus (F_u \cup F_{u'})$  are valid for the edge  $uu'$ , then by Fact 2.5, for each  $\gamma \in C \setminus (F_u \cup F_{u'})$ , there exists an  $(\alpha, \gamma, u, u')$ -critical path. Since  $c'(ua) = \alpha$ , we have all the critical paths passing through the vertex  $a$  and hence  $S_{ua} \subseteq C \setminus (F_u \cup F_{u'})$ . This implies that  $|S_{ua}| \geq |C \setminus (F_u \cup F_{u'})| \geq \Delta + 12 - (1 + \Delta - 1 - 1) = 13$ , a contradiction since  $|S_{ua}| = 1$ . Thus we have a valid colour for the edge  $uu'$ , a contradiction to the assumption that  $G$  is a counter-example.

This final contradiction completes the proof of Theorem 2.1.

**3. Planar graphs of girth at least 5.** The aim of this section is to prove Conjecture 1.2 for planar graphs of girth at least 5. Actually, we prove the conjecture for a more general class of graphs: the graphs of maximum average degree at most  $10/3$ . The *average degree* of a graph  $G$  is  $Ad(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v) = \frac{2|E(G)|}{|V(G)|}$ . The *maximum average degree* of  $G$  is  $Mad(G) = \max\{Ad(H) \mid H \text{ is a subgraph of } G\}$ . It is well known that the girth and the maximum average degree of a planar graph are related to each other:

PROPOSITION 3.1. *Let  $G$  be a planar graph of girth  $g$ .*

$$\text{Mad}(G) < 2 + \frac{4}{g-2}.$$

THEOREM 3.2. *Let  $\Delta \geq 19$  and  $G$  be a graph with maximum degree at most  $\Delta$  and maximum average degree less than  $\frac{10}{3}$ . Then  $\chi'_a(G) \leq \Delta$ .*

Theorem 3.2 and Proposition 3.1 immediately yield the following.

COROLLARY 3.3. *Let  $\Delta \geq 19$  and  $G$  be a planar graph with maximum degree at most  $\Delta$  and girth at least 5. Then  $\chi'_a(G) \leq \Delta$ .*

More generally than Theorem 3.2, we show the following.

THEOREM 3.4. *For any  $\epsilon > 0$ , there exists an integer  $\Delta_\epsilon$  such that every graph  $G$  with maximum degree at most  $\Delta$  with  $\Delta \geq \Delta_\epsilon$  and maximum average degree less than  $4 - \epsilon$  is acyclically  $\Delta$ -edge-colourable.*

In order to prove Theorems 3.2 and 3.4, we first establish some properties of  $\Delta$ -minimal graphs which are graphs with maximum degree at most  $\Delta$ , not acyclically  $\Delta$ -edge-colourable but such that every proper subgraph is. Then, by the Discharging Method, we deduce that such a graph has maximum average degree at least  $4 - \epsilon$  (resp.  $10/3$ ) if  $\Delta$  is at least  $\Delta_\epsilon$  (resp. 19). We will first prove, in Subsection 3.2, Theorem 3.4 for its discharging procedure is simpler because we only establish the existence of  $\Delta_\epsilon$  and make no attempt to minimize it. We then show Theorem 3.2 in Subsection 3.3.

A vertex of degree  $i$  is called an  $i$ -vertex and an  $i$ -neighbour of a vertex  $v$  is a neighbour of  $v$  having degree  $i$ .

**3.1. Properties of  $\Delta$ -minimal graphs.** PROPOSITION 3.5. *A  $\Delta$ -minimal graph  $G$  is 2-connected. In particular,  $\delta(G) \geq 2$ .*

*Proof.* If  $G$  is not connected, it is the disjoint union of  $G_1$  and  $G_2$ . Both  $G_1$  and  $G_2$  admits an acyclic  $\Delta$ -edge-colouring by minimality of  $G$ . The union of these two edge-colourings is an acyclic  $\Delta$ -edge-colouring of  $G$ .

Suppose now that  $G$  has a cutvertex  $v$ . Let  $C_i$ , for  $1 \leq i \leq p$  be the components of  $G - v$  and  $G_i$  the graph induced by  $C_i \cup \{v\}$ . By minimality of  $G$ , all the  $G_i$  admit an acyclic  $\Delta$ -edge-colouring. Moreover, free to permute the colours we may assume that two edges incident to  $v$  get different colours. Hence the union of these edge-colourings is an acyclic  $\Delta$ -edge-colouring of  $G$  because any cycle of  $G$  is entirely contained in one of the  $G_i$ .  $\square$

PROPOSITION 3.6. *Let  $G$  be a  $\Delta$ -minimal graph. For every vertex  $v \in V(G)$ ,  $\sum_{u \in N(v)} d(u) \geq \Delta + 1$ .*

*Proof.* Suppose by way of contradiction that there is a vertex  $v$  such that  $\sum_{u \in N(v)} d(u) \leq \Delta$ . Let  $w$  be a neighbour of  $v$ . By minimality of  $G$ ,  $G \setminus vw$  admits an acyclic edge-colouring with  $\Delta$  colours. Now colour  $vw$  with a colour distinct from the ones of the edges incident to a neighbour of  $v$ . This is possible as there are at most  $\Delta - 1$  such edges distinct from  $vw$ . Doing so we clearly obtain a proper edge-colouring. Let us now show that there is no bicoloured cycle. A cycle that does not contain  $vw$  has edges of at least three colours as the edge-colouring of  $G$  was acyclic and a cycle containing  $vw$  must contain an edge  $vu$  and an edge  $tu$  with  $u \in N(v) \setminus \{w\}$ . By construction, the colours of  $tu$ ,  $uv$  and  $vw$  are distinct.  $\square$

A *thread* is a path of length two whose internal vertex has degree 2.

PROPOSITION 3.7. *Let  $k \geq 2$  be an integer and  $G$  a  $\Delta$ -minimal graph. In  $G$ , a  $\Delta$ -vertex is the end of at most  $k$  threads whose other endvertex has degree at most  $k$ .*

To prove this proposition we need the following lemma.

LEMMA 3.8. *Let  $H = ((A, B), E)$  be a bipartite graph with  $|A| = |B| = q$  such that for any vertex  $a \in A$   $d(a) = 1$  and let  $K_{A,B}$  be the complete bipartite graph with bipartition  $(A, B)$ . If at least 3 vertices of  $B$  of degree at least one in  $H$  then there exists a perfect matching  $M$  of  $K_{A,B}$  such that the bipartite graph  $((A, B), E \cup M)$  has girth at least 6.*

*Proof.* Let  $m$  be the number of vertices of  $B$  of degree at least one. Let  $b_1, \dots, b_q$  be the vertices of  $B$  with  $d(b_i) \geq 1$  if  $i \leq m$  and  $d(b_i) = 0$  otherwise. And let  $a_1, \dots, a_q$  be the vertices of  $A$  with  $a_i b_i \in E$  for all  $1 \leq i \leq m$ . If  $m \geq 3$ , let  $M = \{a_i b_{i+1} \mid 1 \leq i < m\} \cup \{a_m b_1\} \cup \{a_i b_i \mid m < i \leq q\}$ . Then the unique cycle in  $((A, B), E \cup M)$  is  $C = (a_1, b_2, a_2, b_3, \dots, a_{m-1}, b_m, a_1)$ . It has length  $2m \geq 6$ .  $\square$

*Proof.* [of Proposition 3.7] Suppose for a contradiction that there is a  $\Delta$ -vertex  $u$  with  $q = k + 1$  threads  $uv_i w_i$ ,  $1 \leq i \leq q$ , such  $d(w_i) \leq k$ . Note that  $q \geq 3$ .

Set  $A = \{v_1, \dots, v_q\}$ . By Proposition 3.5,  $w_i \notin A$  for all  $1 \leq i \leq q$ . By minimality of  $G$ ,  $G - A$  admits an acyclic  $\Delta$ -edge-colouring.

Let us first extend it to the  $v_i w_i$  as follows. Let  $F$  be the set of colours assigned to the edges incident to  $u$  and to no vertex of  $A$  and for  $1 \leq i \leq q$  let  $F_i$  be the set of colours assigned to the edges incident to  $w_i$  (and distinct from  $v_i w_i$ ). Then  $|F| = \Delta - q$  and  $|F_i| \leq k - 1$ . For all  $1 \leq i \leq q$ , let  $S_i$  be the set of colours not in  $F \cup F_i$ . Since  $|F| + |F_i| = \Delta - q + k - 1 = \Delta - 2$  then  $|S_i| \geq 2$ .

. Assume first that  $|\bigcup_{i=1}^q S_i| \geq 3$ , then one can assign to each  $v_i w_i$  a colour in  $S_i$  in such a way that at least 3 colours appear on such edges and that different colours appear on  $v_i w_i$  and  $v_j w_j$  if  $w_i = w_j$ . We will now colour the edges  $uv_i$  for  $1 \leq i \leq q$ . Therefore let  $H_1 = ((A, B), E_1)$  be the bipartite graph with  $B$  the set of  $q$  colours  $\{b_1, \dots, b_q\}$  not in  $F$  and in which  $v_i$  is adjacent to  $b_j$  if  $c(v_i w_i) = b_j$ . As long as some  $v_i$  has degree 0 then add an edge between  $a_i$  and an isolated  $b_j$  to obtain a bipartite graph  $H_2 = ((A, B), E_2)$ . Because at least three colours appear on the  $v_i w_i$ , the graph  $H_2$  fulfils the hypothesis of Lemma 3.8. So there exists a perfect matching  $M$  of  $K_{A,B}$  such that  $((A, B), E_2 \cup M)$  has girth at least 6. For  $1 \leq i \leq q$ , assign to each  $uv_i$  the colour to which  $v_i$  is linked in  $M$ .

Let us now prove that this edge-colouring of  $G$  is acyclic. It is obvious that it is proper since  $v_i$  is not linked to  $c(v_i w_i)$  in  $M$ . Let us now prove that it is acyclic. Let  $C$  be a cycle of  $G$ . If it contains no vertex of  $A$ , then it contains edges of three different colours because the edge-colouring of  $G - A$  is acyclic. Suppose now that  $C$  contains a unique vertex of  $A$ , say  $v_i$ . Then  $C$  contains  $w_i v_i$ ,  $v_i u$  and  $ut$  with  $t$  a neighbour of  $u$  not in  $A$ . Then  $c(ut) \in F$ , so by construction,  $c(w_i v_i) \neq c(ut)$ . Hence the colours of  $w_i v_i$ ,  $v_i u$  and  $ut$  are distinct. Suppose finally that  $C$  contains two vertices of  $A$ , say  $v_i$  and  $v_j$ . Then  $C$  contains  $w_i v_i$ ,  $v_i u$ ,  $w_j v_j$  and  $v_j u$ . Since  $((A, B), E_2 \cup M)$  has girth at least 6, either  $c(v_i u) \neq c(w_j v_j)$  or  $c(v_j u) \neq c(w_i v_i)$ . In both cases,  $C$  has edges of three different colours.

. Assume now that  $|\bigcup_{i=1}^q S_i| < 3$ . Then all the  $S_i$  are equal and of cardinality 2, say  $S_i = \{a, b\}$  for all  $1 \leq i \leq q$ . Hence all the  $F_i$  are the same of cardinality  $k - 1$  and disjoint from  $F$ . Observe that this can happen only if all the  $w_i$  are distinct. Let us denote by  $f_1, \dots, f_{k-1}$  the elements of the  $F_i$ . Let us set  $c(v_i w_i) = a$  for  $1 \leq i \leq k$ ,  $c(v_q w_q) = b$ ,  $c(uv_i) = f_i$  for  $1 \leq i \leq k - 1$ ,  $c(uv_k) = b$  and  $c(uv_{k+1}) = a$ . It is easy to check that the obtained edge-colouring is an acyclic edge-colouring of  $G$ .  $\square$

**PROPOSITION 3.9.** *Let  $k$  and  $l$  be two positive integers and  $G$  a  $\Delta$ -minimal graph. In  $G$ , a  $(\Delta - l)$ -vertex is the end of at most  $k - 1 - l$  threads whose other endvertex has degree at most  $k$ .*

To prove this proposition we need the following lemma.

**LEMMA 3.10.** *Let  $H = ((A, B), E)$  be a bipartite graph with  $q = |A| < |B|$  such that for any vertex  $a \in A$   $d(a) = 1$  and  $K_{A,B}$  be the complete bipartite graph with bipartition  $(A, B)$ .*

*Then there exists a matching  $M$  of  $K_{A,B}$  saturating  $A$  such that the bipartite graph  $((A, B), E \cup M)$  has no cycle.*

*Proof.* Let  $q' = |B|$ . Let  $b_1, \dots, b_{q'}$  be the vertices of  $B$  with  $d(b_i) \geq 1$  if  $i \leq m$  and  $d(b_i) = 0$  otherwise. And let  $a_1, \dots, a_q$  be the vertices of  $A$  with  $a_i b_i \in E$  for all  $1 \leq i \leq m$ . Let  $M = \{a_i b_{i+1} \mid 1 \leq i \leq q\}$ . This is well-defined since  $q' > q$ . Then  $((A, B), E \cup M)$  has no cycle.  $\square$

*Proof.* [of Proposition 3.9]. Suppose for a contradiction that there is a  $(\Delta - l)$ -vertex  $u$  with  $q = k - l$  threads  $uv_i w_i$ ,  $1 \leq i \leq q$ , such  $d(w_i) \leq k$ .

Set  $A = \{v_1, \dots, v_q\}$ . By minimality of  $G$ ,  $G - A$  admits an acyclic  $\Delta$ -edge-colouring. Let us first extend it to the  $v_i w_i$  as follows. Let  $F$  be the set of colours assigned to the edges incident to  $u$  and to no vertex of  $A$  and for  $1 \leq i \leq q$  let  $F_i$  be the set of colours assigned to the edges incident to  $w_i$  (and

distinct from  $v_i w_i$ ). Then  $|F| = \Delta - l - q$  and  $|F_i| \leq k - 1$ .

For all  $1 \leq i \leq q$  colour  $v_i w_i$  with a colour not in  $F \cup F_i$  and distinct from the colours. This is possible since  $|F| + |F_i| = \Delta - l - q + k - 1 = \Delta - 1$ .

We will now colour the edges  $uv_i$  for  $1 \leq i \leq q$ . Therefore let  $H_1 = ((A, B), E_1)$  be the bipartite graph with  $B$  the set of  $q + j$  colours  $\{b_1, \dots, b_{q+j}\}$  not in  $F$  and in which  $v_i$  is adjacent to  $b_j$  if  $c(v_i w_i) = b_j$ . As long as some  $v_i$  has degree 0 then add an edge between  $a_i$  and an isolated  $b_j$  to obtain a bipartite graph  $H_2 = ((A, B), E_2)$ . Then  $H_2$  fulfils the hypothesis of Lemma 3.10 so there exists a perfect matching  $M$  of  $K_{A,B}$  such that  $((A, B), E_2 \cup M)$  has no cycle. For  $1 \leq i \leq q$ , assign to each  $uv_i$  the colour to which  $v_i$  is linked in  $M$ .

In the same way as in the proof of Proposition 3.7, one shows that the obtained edge-colouring is acyclic.  $\square$

**3.2. Proof of Theorem 3.4.** LEMMA 3.11. *Let  $\epsilon > 0$ . There exists  $\Delta_\epsilon$  such that if  $\Delta \geq \Delta_\epsilon$  then any  $\Delta$ -minimal graph has average degree at least  $4 - \epsilon$ .*

*Proof.* The result for  $\epsilon = \frac{1}{2}$  implies the result for larger values of  $\epsilon$ . Hence we assume that  $\epsilon \leq \frac{1}{2}$ . Let us assign an initial charge of  $d(v)$  to each vertex  $v \in V(G)$  Set  $d_\epsilon = \lceil \frac{\Delta}{\epsilon} - 2 \rceil$ .

We perform the following discharging rules.

**R1:** for  $4 \leq d < d_\epsilon$ , every  $d$ -vertex sends  $a(d) = 1 - \frac{4-\epsilon}{d}$  to each neighbour.

**R2:** for  $d_\epsilon \leq d \leq \Delta + 1 - d_\epsilon$  then every  $d$ -vertex sends  $1 - \frac{\epsilon}{2}$  to each neighbour.

**R3:** for  $\Delta + 2 - d_\epsilon \leq d \leq \Delta$  then every  $d$ -vertex sends

- $1 - \epsilon$  to each 3-neighbour;
- $2 - \epsilon$  to each 2-neighbour whose second neighbour has degree 2 or 3;
- $b(d) = 2 - \epsilon - a(d)$  to each 2-neighbour whose second neighbour has degree  $d$  with  $4 \leq d < d_\epsilon$ ;
- $1 - \frac{\epsilon}{2}$  to each 2-neighbour whose second neighbour has degree  $d \geq d_\epsilon$ .

Let us now check that every vertex  $v$  has final charge  $f(v)$  at least  $4 - \epsilon$ .

If  $v$  is a 2-vertex then let  $u$  and  $w$  be its two neighbours with  $d(u) \leq d(w)$ . If  $d(u) \leq 3$  then  $d(w) \geq \Delta - 2$  by Proposition 3.6. Hence  $v$  receives  $2 - \epsilon$  from  $w$  by R3, so  $f(v) \geq 2 + 2 - \epsilon = 4 - \epsilon$ . If  $4 \leq d(u) < d_\epsilon$  then  $d(w) > \Delta + 1 - d_\epsilon$  by Proposition 3.6. Hence  $v$  receives  $a(d)$  from  $u$  by R2 and  $b(d)$  from  $w$  by R3. So  $f(v) = 4 - \epsilon$ . If  $d(u) \geq 10$  then  $v$  receives  $1 - \frac{\epsilon}{2}$  from  $u$  and  $1 - \frac{\epsilon}{2}$  from  $w$  by R3. So  $f(v) = 4 - \epsilon$ .

Suppose that  $v$  is a 3-vertex. Then by Proposition 3.6 it has at least two ( $\geq 8$ )-neighbours. Hence it receives at least  $2 \times 1/2$  by R1, R2 or R3 because  $\epsilon \leq \frac{1}{2}$ . So  $f(v) \geq 4$ .

Suppose  $4 \leq d(v) < d_\epsilon$ . Then  $v$  sends  $d(v)$  times  $1 - \frac{4-\epsilon}{d(v)}$  so  $f(v) \geq 4 - \epsilon$ .

Suppose  $d_\epsilon \leq d(v) \leq \Delta + 1 - d_\epsilon$ . Then  $v$  sends at most  $d(v)$  times  $1 - \frac{\epsilon}{2}$  so  $f(v) \geq d(v) \times \frac{\epsilon}{2} \geq 4 - \epsilon$ .

Suppose now that  $d(v) \geq \Delta + 2 - d_\epsilon$ . Then by Propositions 3.7 and 3.9, the most  $v$  can send is when it has three 2-neighbours with second neighbour of degree at most 3, one 2-neighbour with second neighbour of degree  $d$  for all  $4 \leq d \leq d_\epsilon - 1$  and  $\Delta - d_\epsilon + 1$  2-neighbours with second neighbour of degree at least  $d_\epsilon$ . Hence

$$\begin{aligned} f(v) &\geq \Delta + 2 - d_\epsilon - 3(2 - \epsilon) - \sum_{d=4}^{d_\epsilon-1} b(d) - (\Delta - d_\epsilon + 1)(1 - \frac{\epsilon}{2}) \\ &\geq \Delta \frac{\epsilon}{2} - S_\epsilon \end{aligned}$$

with  $S_\epsilon = d_\epsilon - 2 + 3(2 - \epsilon) + \sum_{d=4}^{d_\epsilon-1} b(d) - (1 - \frac{\epsilon}{2})(d_\epsilon - 1)$ . Setting  $\Delta_\epsilon = \lceil \frac{2}{\epsilon}(S_\epsilon + 4 - \epsilon) \rceil$ , if  $\Delta \geq \Delta_\epsilon$ ,  $f(v) \geq 4 - \epsilon$ .  $\square$

*Proof.* [of Theorem 3.4] If Theorem 3.4 were false, then a minimum counterexample  $G$  would be a  $\Delta$ -minimum graph. So by Lemma 3.11, its average degree would be at least  $4 - \epsilon$ , a contradiction.  $\square$

**3.3. Proof of Theorem 3.2.** Lemma 3.11 for  $\epsilon = 2/3$  yields that for  $\Delta \geq \Delta_{2/3}$ , a  $\Delta$ -minimal graph  $G$  satisfies  $Mad(G) \geq Ad(G) \geq 10/3$ . The value of  $\Delta_{2/3}$  given by the proof of Lemma 3.11 is 49. We now show that it could be decreased to 19.

LEMMA 3.12. *Let  $\Delta \geq 19$  and  $G$  be a  $\Delta$ -minimal graph. Then  $Mad(G) \geq Ad(G) \geq 10/3$ .*

*Proof.* Let us assign an initial charge of  $d(v)$  to each vertex  $v \in V(G)$  and perform the following discharging rules.

- R1:** every 4-vertex sends  $4/9$  to each of its ( $\leq 3$ )-neighbours;
- R2:** every 5-vertex sends  $7/12$  to each 2-neighbour and  $1/3$  to each 3-neighbour;
- R3:** for  $6 \leq d \leq 9$ , every  $d$ -vertex sends  $1 - 10/3d$  to each neighbour.
- R4:** for  $10 \leq d \leq \Delta - 9$  then every  $d$ -vertex sends  $2/3$  to each neighbour.
- R5:** for  $\Delta - 8 \leq d \leq \Delta$  then every  $d$ -vertex sends
  - $2/3$  to each  $d$ -neighbour with  $3 \leq d \leq 5$ ;
  - $4/3$  to each 2-neighbour whose second neighbour has degree 2 or 3;
  - $8/9$  to each 2-neighbour whose second neighbour has degree 4;
  - $9/12$  to each 2-neighbour whose second neighbour has degree 5;
  - $1/3 + 10/3d$  to each 2-neighbour whose second neighbour has degree  $d$  with  $6 \leq d \leq 9$ ;
  - $2/3$  to each 2-neighbour whose second neighbour has degree  $d \geq 10$ .

Let us now check that every vertex  $v$  has final charge  $f(v)$  at least  $\frac{10}{3}$ .

If  $v$  is a 2-vertex then let  $u$  and  $w$  be its two neighbours with  $d(u) \leq d(w)$ . If  $d(u) \leq 3$  then  $d(w) \geq \Delta - 2$  by Proposition 3.6. Hence  $v$  receives  $4/3$  from  $w$  by R5, so  $f(v) \geq 2 + 4/3 = 10/3$ . If  $d(u) = 4$  then  $d(w) \geq \Delta - 3$  by Proposition 3.6. Hence  $v$  receives  $4/9$  from  $u$  by R1 and  $8/9$  from  $w$  by R5. So  $f(v) = 10/3$ . If  $d(u) = 5$  then  $d(w) \geq \Delta - 4$  by Proposition 3.6. Hence  $v$  receives  $7/12$  from  $u$  by R2 and  $9/12$  from  $w$  by R5. So  $f(v) = 10/3$ . If  $6 \leq d(u) \leq 9$  then  $d(w) \geq \Delta - 8$  by Proposition 3.6. Hence  $v$  receives  $1 - 10/3d$  from  $u$  by R3 and  $1/3 + 10/3d$  from  $w$  by R5. So  $f(v) = 10/3$ . If  $d(u) \geq 10$  then  $v$  receives  $2/3$  from  $u$  by R4 and  $2/3$  from  $w$  by R5. So  $f(v) = 10/3$ .

Suppose that  $v$  is a 3-vertex. Then, since  $\Delta \geq 10$ , by Proposition 3.6 it has either a ( $\geq 5$ )-neighbour or two 4-neighbours. Hence it receives either at least  $1/3$  by R2, R3, R4 or R5, or  $2 \times 4/9 \geq 1/3$  by R1. In both cases,  $f(v) \geq 3 + 1/3 = 10/3$ .

Suppose that  $v$  is a 4-vertex. Then, since  $\Delta \geq 18$ , by Proposition 3.6, it has either three ( $\leq 3$ )-neighbours and one ( $\geq 10$ )-neighbour or at most two ( $\leq 3$ )-neighbours. In the first case, it sends  $4/9$  to each of its 3-neighbours and receives  $2/3$  from its ( $\geq 10$ )-neighbour. So  $f(v) \geq 4 - 3 \times \frac{4}{9} + \frac{2}{3} = 10/3$ . In the second case, it sends  $4/9$  to at most 2 neighbours. So  $f(v) \geq 4 - 2 \times \frac{4}{9} > 10/3$ .

Suppose that  $v$  is a 5-vertex.

Assume first that  $v$  has at most three ( $\leq 3$ )-neighbours. If it has at least one (3)-neighbour it sends at most  $3/2$  so  $f(v) \geq 5 - 3/2 > 10/3$ . If not it has three 2-neighbours. Let  $u_1$  and  $u_2$  be the two ( $\geq 4$ )-neighbours of  $v$ . By Proposition 3.6,  $d(u_1) + d(u_2) \geq 11$  since  $\Delta \geq 16$ . Hence one of these two vertices is a ( $\geq 6$ )-vertex and it sends at least  $4/9$  to  $u$ . Hence  $f(v) \geq 5 + 4/9 - 7/4 > 10/3$ .

Assume now that  $v$  has at least four ( $\leq 3$ )-neighbours. Let  $i$  be the number of 2-neighbours of  $v$ . Then by Proposition 3.6,  $v$  has exactly  $4 - i$  3-neighbours and its fifth neighbour has degree at least  $6 + i$  since  $\Delta \geq 17$ . Hence  $f(v) \geq 5 - i \cdot \frac{7}{12} - (4 - i) \frac{1}{3} + 1 - \frac{10}{3(6+i)} > 10/3$ .

Suppose  $6 \leq d(v) \leq 9$ . Then  $v$  sends  $d(v)$  times  $1 - 10/3d(v)$  so  $f(v) \geq d(v) - d(v)(1 - 10/3d) = 10/3$ .

Suppose  $10 \leq d(v) \leq \Delta - 10$ . Then  $v$  sends at most  $d(v)$  times  $2/3$  so  $f(v) \geq d(v)(1 - 2/3) \geq 10/3$ .

Suppose that  $d(v) = \Delta - l$  for  $1 \leq l \leq 7$ . By Proposition 3.9,  $v$  is incident to at most  $\Delta - l - 1$  threads so its has at least one ( $\geq 3$ )-neighbour to which it sends at most  $2/3$ . Moreover the most it can send is when it has exactly one 2-neighbour with second neighbour of degree  $d$  for each  $l + 2 \leq d \leq 9$  and

$\Delta - 9$  2-neighbours with second neighbour of degree at least 10. Hence its final charge is

$$\begin{aligned} f(v) &\geq \Delta - l - \left( (\Delta - 8) \frac{2}{3} + \sum_{d=l+2}^9 s(d) \right) \\ &\geq \frac{1}{3} \Delta + \frac{16}{3} - \left( l + \sum_{d=l+2}^9 s(d) \right) \end{aligned}$$

with  $s(3) = 4/3$ ,  $s(4) = 8/9$ ,  $s(5) = 9/12$  and  $s(d) = 1/3 + 10/3d$  for  $6 \leq d \leq 9$ . Since  $s(3) > 1$  and  $s(d) < 1$  when  $d \geq 4$ , then  $l + \sum_{d=l+2}^9 s(d)$  is minimum when  $l = 2$ . Hence

$$\begin{aligned} f(v) &\geq \frac{1}{3} \Delta + \frac{16}{3} - \left( 2 + \sum_{d=4}^9 s(d) \right) \\ &\geq \frac{1}{3} \Delta + \frac{61}{36} - \frac{10}{3} \sum_{d=6}^9 \frac{1}{d} \\ &\geq \frac{1}{3} \Delta + \frac{61}{36} - \frac{10}{3} \times \frac{275}{504} \geq \frac{10}{3} \end{aligned}$$

because  $\Delta \geq 11$ .

Suppose  $d(v) = \Delta$ . By Proposition 3.7, the most it can send is when it has three 2-neighbours with second neighbour of degree at most 3, exactly one 2-neighbour with second neighbour of degree  $d$  for  $4 \leq d \leq 9$  and  $\Delta - 9$  2-neighbours with second neighbour of degree at least 10. In this case it sends

$$\begin{aligned} 3 \times \frac{4}{3} + \frac{8}{9} + \frac{9}{12} + \sum_{d=6}^9 \left( \frac{1}{3} + \frac{10}{3d} \right) + (\Delta - 9) \frac{2}{3} &= \frac{2}{3} \Delta + \frac{35}{36} + \frac{10}{3} \sum_{d=6}^9 \frac{1}{d} \\ &= \frac{2}{3} \Delta + \frac{35}{36} + \frac{10}{3} \times \frac{275}{504} \\ &\leq \Delta - \frac{10}{3} \end{aligned}$$

because  $\Delta \geq 19$ . Hence  $f(v) \geq \frac{10}{3}$ .

Now  $Ad(G) = \frac{1}{|V|} \sum_{v \in V(G)} d(v) = \frac{1}{|V|} \sum_{v \in V(G)} f(v) \geq \frac{10}{3}$ .  $\square$

*Proof.* [ of Theorem 3.2] If Theorem 3.2 would be false, a minimum counterexample  $G$  would be a  $\Delta$ -minimum graph. So by Lemma 3.12, its average degree is at least  $10/3$ , a contradiction.  $\square$

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