On the Grundy number of graphs with few P4's

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On the Grundy number of graphs with few P_4's[☆]

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Abstract

The Grundy number of a graph G is the largest number of colors used by any execution of the greedy algorithm to color G. The problem of determining the Grundy number of G is polynomial if G is a P_4-free graph and NP-hard if G is a P_5-free graph. In this article, we define a new class of graphs, the fat-extended P_4-laden graphs, and we show a polynomial time algorithm to determine the Grundy number of any graph in this class. Our class intersects the class of P_5-free graphs and strictly contains the class of P_4-free graphs. More precisely, our result implies that the Grundy number can be computed in polynomial time for any graph of the following classes: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden, which are all strictly contained in the fat-extended P_4-laden class.

Keywords: Graph Theory, Grundy number, P_4-classes, Modular decomposition.

1. Introduction

Given a graph $G = (V, E)$, a vertex coloring of G is a mapping c from the vertex set of G to a set of positive integers (colors) in such a way that for every pair of adjacent vertices u and v of G, $c(u) \neq c(v)$. A coloring $c : V(G) \longmapsto \{1, \ldots, k\}$ is a k-coloring of G. Since the subset of vertices assigned to the same color induces a stable set of G, a k-coloring can be also seen as a partition $c = \{S_1, \ldots, S_k\}$ of $V(G)$ into stable sets such that each S_i, $1 \leq i \leq k$, contains the vertices colored i. We say that a color i *occurs* in a set of vertices if there is some vertex of this set colored i. The smallest number k for which G admits a k-coloring is the chromatic number $\chi(G)$ of G. Determining the chromatic number of a graph is a NP-hard problem [1].

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In the *on-line* version of the problem, the vertices of the input graph are presented to a coloring algorithm one at a time in some arbitrary order. The algorithm must choose a color for each vertex, based only on the colors assigned to the already-processed vertices. The *on-line* chromatic number of a graph G is the minimum number of colors needed to color on-line the vertices of G when they are given in the worst possible order [2, 3]. Several on-line coloring algorithms have been designed. The most popular one is the *greedy algorithm*. Given a graph $G = (V, E)$ and an order $\theta = v_1, \dots, v_n$ over V, the greedy algorithm assigns to v_i the minimum positive integer that was not already assigned to its neighborhood in the set $\{v_1, \dots, v_{i-1}\}$. A greedy coloring is a coloring obtained by this algorithm. The maximum number of colors required by the greedy algorithm to color a graph G, over all the orders θ of $V(G)$, is the *Grundy number* of G and it is denoted by $\Gamma(G)$. Observe that the Grundy number of a graph is an upper bound for its chromatic number as well as its on-line chromatic number.

Determining the Grundy number is NP-hard for general graphs [4] and also for complements of bipartite graphs [5] and, as a consequence, for P_5-free graphs, since every complement of a bipartite graph is P_5-free. In fact, given a graph G and an integer r it is a $coNP$-complete problem to decide if $\Gamma(G) \leq \chi(G) + r$ or if $\Gamma(G) \leq r \times \chi(G)$ or if $\Gamma(G) \leq c \times \omega(G)$ [6, 5], where $\omega(G)$ stands for the size of a maximum clique of G. However, there are polynomial time algorithms to determine the Grundy number of the following classes of graphs: P_4-free graphs [2], trees [7], k-partial trees [8] and hypercubes [9]. Moreover, given a graph $G = (V, E)$ and an integer k, there is an algorithm to determine if $\Gamma(G) \geq k$ with complexity $\mathcal{O}(n^{2^{k-1}})$ [10].

In this article, we introduce a new class of graphs, the *fat-extended P_4-laden graphs*, and we present a polynomial time algorithm to calculate the Grundy number of any graph of this class, using modular decomposition. Our class intersects the class of the P_5-free graphs class and strictly contains the class of P_4-free graphs. More precisely, our result implies that the Grundy number can be determined in polynomial time for any graph of the following classes: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden, which are all strictly contained in the fat-extended P_4-laden class.

The paper is organized as follows. In Section 2, we introduce some basic concepts related to modular decomposition, besides other simple definitions. In Section 3, we recall the definition of extended P_4-laden graphs and we define our new class of graphs. We present the algorithm and we prove its correctness and complexity in Section 4. Finally, we comment the results in Section 5.

2. Preliminaries

Let $G = (V, E)$ be a graph and S a subset of $V(G)$. We denote by $G[S]$ the subgraph of G induced by S and denote by $N_G(v)$ the set of neighbors of a vertex v in G (or just $N(v)$ when G is clear in the context). We say that $M \subseteq V(G)$ is a *module* of a graph G if, for every vertex w of $V \backslash M$, either w is adjacent to all the vertices of M or w is adjacent to none of them. The sets V and $\{x\}$, for every $x \in V$, are *trivial modules*, the latest being called a *singleton* module.

A graph is *prime* if all its modules are trivial. We say that M is a *strong module* of G if, for every module M' of G, either $M' \cap M = \emptyset$ or $M \subset M'$ or $M' \subset M$. The *modular decomposition* of a graph G is a decomposition of G that associates with G a unique *modular decomposition tree* $T(G)$. The modular decomposition tree of G, $T(G)$, is a rooted tree where the leaves are the vertices of G, and such that any maximal set of its leaves having the same least common ancestor v is a strong module of G, which is denoted by $M(v)$.

Let r be an internal node of $T(G)$ and $V(r) = \{r_1, \ldots, r_k\}$ be the set of children of r in $T(G)$. If $G[M(r)]$ is disconnected, then r is called a *parallel* node and $G[M(r_1)], \ldots, G[M(r_k)]$ are its components. If $\bar{G}[M(r)]$ is disconnected then r is called a *series* node and $\bar{G}[M(r_1)], \ldots, \bar{G}[M(r_k)]$ are the components of $\bar{G}[M(r)]$. Finally, if both graphs $G[M(r)]$ and $\bar{G}[M(r)]$ are connected, then r is called a *neighborhood* node and $\{M(r_1), \ldots, M(r_k)\}$ is the unique set of maximal strong submodules of $M(r)$.

The *quotient* graph of $G[M(r)]$, denoted by $G(r)$, is $G[\{v_1, \ldots, v_k\}]$, where $v_i \in M(r_i)$, for $1 \leq i \leq k$. We say that r is a *fat* node, if $M(r)$ is not a singleton module.

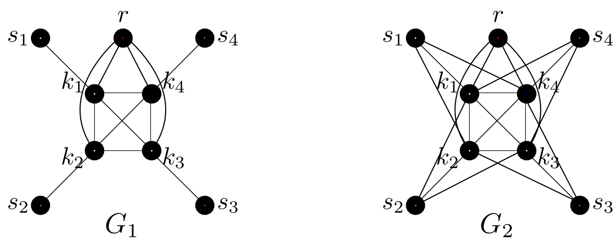

Figure 1: examples of thin (G_1) and thick (G_2) spiders with partition $S = \{s_1, s_2, s_3, s_4\}$, $K = \{k_1, k_2, k_3, k_4\}$ and $R = \{r\}$.

A graph is a *spider* (see Figure 1) if its vertex set can be partitioned into three sets S, K and R in such a way that S is a stable set, K is a clique, all the vertices of R are adjacent to all the vertices of K and to none of the vertices of S and there exists a bijection $f : S \to K$ such that, for all $s \in S$, either $N(s) = f(s)$ (and it is a *thin spider*) or $N(s) = K - f(s)$ (and it is a *fat spider*).

A graph $G = (V = S \cup K, E)$ is *split* if its vertex set can be partitioned into a stable set S and a clique K. Observe that the spiders of Figure 1 are also split graphs, since R is a clique and by consequence $V = (S, K \cup R)$ is a partitioning of the vertices of both spiders into a stable set and a clique. Alternately, the vertices of a split graph $G = (V = S \cup K, E)$ can also be partitioned into three disjoint sets $S'(G)$, $K'(G)$ and $R'(G)$, such that every vertex of S which looses at least one vertex in K belongs to $S'(G)$, $K'(G) \subseteq K$ is the neighborhood of the vertices in $S'(G)$ and $R'(G) = V \backslash S'(G) \cup K'(G)$ (see Figure 2). It is well-known that a graph is split if and only if it is $\{C_5, C_4, \bar{C}_4\}$-free [11]. A *pseudo-split* graph is a $\{C_4, \bar{C}_4\}$-free graph.

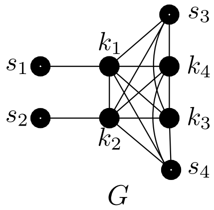

Figure 2: example of split graph G with partitioning $S'(G) = \{s_1, s_2\}$, $K'(G) = \{k_1, k_2\}$ and $R'(G) = \{s_3, s_4, k_3, k_4\}$.

3. Fat extended P_4-laden graphs

Giakoumakis [12] defined a graph G as *extended P_4-laden graph* if, for all $H \subseteq G$ such that $|V(H)| \leq 6$, the following statement is true: if H contains more than two induced P_4's, then H is a pseudo-split graph. It follows that an extended P_4-laden graph can be completely characterized by its modular decomposition tree, as follows:

Theorem 1. *[12] Let $G = (V, E)$ be a graph, $T(G)$ be its modular decomposition tree and r be any neighborhood node of $T(G)$, with children $r_1, \ldots, r_k$. Then G is extended P_4-laden if and only if $G(r)$ is isomorphic to:*

1. *a P_5 or a $\bar{P}_5$ or a C_5, and each $M(r_i)$, $1 \leq i \leq k$, is a singleton module; or*
2. *a spider $H = (S \cup K \cup R, E)$ and each $M(r_i)$, $1 \leq i \leq k$, is a singleton module, except the one corresponding to R and occasionally another one which may have exactly two vertices; or*
3. *a split graph $H = (S \cup K, E)$, whose modules corresponding to the vertices of S are independent sets and the ones corresponding to the vertices of K are cliques.*

We say that a graph is **fat-extended P_4-laden** if its modular decomposition satisfies Theorem 1, except in the first case, where $G(r)$ is isomorphic to a P_5 or a $\bar{P}_5$ or a C_5, but the maximal strong modules $M(r_i)$, $1 \leq i \leq 5$, of $M(r)$ are not necessarily singleton modules.

Observe that the class of fat-extended P_4-laden graphs contains the class of extended P_4-laden graphs. Figure 3 shows us an example of a fat-extended P_4-laden graph that is not an extended P_4-laden graph.

Consequently, the class of fat-extended P_4-laden graphs strictly contains all the following classes of graphs: P_4-reducible, extended P_4-reducible, P_4-sparse,

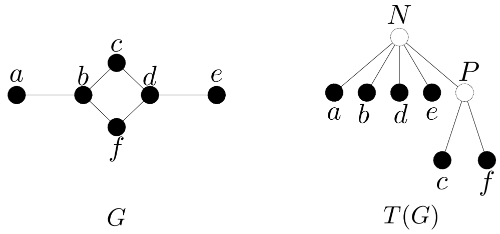

Figure 3: Example of a fat-extended P_4-laden graph which is not an extended P_4-laden graph.

extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden. Notice that these classes are all contained in the class of extended P_4-laden graphs [13].

4. Grundy number on fat-extended P_4-laden graphs

Let $G = (V, E)$ be a fat-extended P_4-laden graph and $T(G)$ be its modular decomposition tree. Since $T(G)$ can be found in linear time [14], we propose an algorithm to determine $\Gamma(G)$ that uses a bottom-up strategy. We know that the Grundy number of the leaves of $T(G)$ is equal to one and we show in this section how to determine the Grundy number of $G[M(v)]$, for each inner node v of $T(G)$, based on the Grundy number of its children.

First, observe that for every series node r of $T(G)$, with children $r_1, \ldots, r_k$, the Grundy number of $G[M(r)]$ is equal to the sum of the Grundy numbers of its children, i.e., $\Gamma(G[M(r)]) = \Gamma(G[M(r_1)]) + \ldots + \Gamma(G[M(r_k)])$. However, if r is a parallel node, the Grundy number of $G[M(r)]$ is the maximum Grundy number among its children, i.e., $\Gamma(G[M(r)]) = \max(\Gamma(G[M(r_1)]), \ldots, \Gamma(G[M(r_k)]))$ [2].

Thus, it remains to prove that the Grundy number of $G[M(r)]$ can be found in polynomial time when r is a neighborhood node of $T(G)$. The following definition will be useful:

Definition 1. *Given two graphs G and H, we say that G' is obtained from G by* replacing *a vertex $v \in V(G)$ by H if $V(G') = \{V(G) \backslash \{v\}\} \cup V(H)$ and $E(G') = \{E(G) \backslash \{uv \mid u \in N_G(v)\}\} \cup E(H) \cup \{uh \mid u \in N_G(v)$ and $h \in H\}$.*

The following result and its proof are a simple generalization of a result due to Asté et al. [6] for the Grundy number of the lexicographic product of graphs.

Proposition 1. *Let G, $H_1, \ldots, H_n$ be disjoint graphs. Let $V(G) = \{v_1, \ldots, v_n\}$ and G' be the graph obtained by replacing $v_i \in V(G)$ by H_i, $1 \leq i \leq n$. Then, for every greedy coloring of G', at most $\Gamma(H_i)$ colors appear in $G'[V(H_i)]$.*

Proof: Consider a greedy coloring c of G' and let $c_1, \ldots, c_p$ be the colors occurring in $G'[V(H_k)]$, for some $k \in \{1, \ldots, n\}$. Denote by S_i, $1 \leq i \leq p$, the stable set formed by the vertices of $G'[V(H_k)]$ colored c_i. Let u_i be a vertex of

S_i. Since c is a greedy coloring, u_i has at least one neighbor w colored c_j, for all $1 \leq j < i \leq p$.

Now, we claim that $w \in G'[V(H_k)]$. By contradiction, suppose that $w \notin G'[V(H_k)]$. So, $w \in V(G') \backslash V(H_k)$. Let $u_j \in G'[V(H_k)]$ be a vertex colored c_j. By Definition 1, once $u_i w$ is an edge, so is $u_j w$, contradicting the assumption that c is a proper coloring. Therefore, $w \in G'[V(H_k)]$. It means that c restricted to $G'[V(H_k)]$, with p colors, is a greedy coloring of $G'[V(H_k)]$ and hence $p \leq \Gamma(G'[H_k]) \leq \Gamma(H_k)$. $\square$

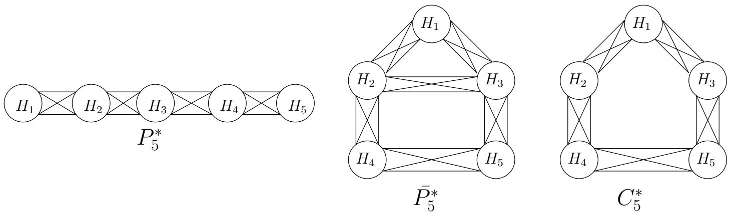

Figure 4: Fat neighborhood nodes.

Let $G = (H_1 \cup \ldots \cup H_5, E)$ be a graph isomorphic to one of the neighborhood nodes depicted in Figure 4. In order to simplify the notation, denote $G[V(H_i)]$ by H_i, $\Gamma(G[H_i])$ by Γ_i and, by θ_i, an order that leads the greedy algorithm to the generation of a greedy coloring of $G[H_i]$ with $\Gamma(G[H_i])$ colors, $i \in \{1, \ldots, 5\}$.

Without loss of generality, we consider, in what follows, that the adjacency between the fat nodes are as depicted in Figure 4.

Lemma 1. *Given the Grundy numbers of the graphs $H_1, \ldots, H_5$, the Grundy number of a $P_5^* = (H_1 \cup \ldots \cup H_5, E)$ can be found in constant time.*

Proof: Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of a P_5^* with $\Gamma(P_5^*)$ colors. So, by definition, each vertex $v \in S_i$ has a neighbor $u \in S_j$, for all $j < i$, $i, j \in \{1, \ldots, k\}$. Let us check all the possible locations of a vertex v colored $\Gamma(G) = k$ in a greedy coloring of G with the maximum number of colors.

1. If there is a vertex $v \in H_1$ colored k, then $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2$.
   In this case, since $N(v) \subseteq V(H_1) \cup V(H_2)$ and $N(v)$ intersects all the stable sets $S_1, \ldots, S_{k-1}$, we have that $\Gamma(P_5^*)$ colors occur in $G[V(H_1) \cup V(H_2)]$. Therefore, by Proposition 1, $k = \Gamma(P_5^*) \leq \Gamma_1 + \Gamma_2$. On the other hand, any ordering over $V(P_5^*)$ that starts by θ_1, followed immediately by θ_2, makes the greedy algorithm generate a greedy coloring of P_5^* with at least $\Gamma_1 + \Gamma_2$ colors.
2. If there is a vertex $v \in V(H_5)$ colored k, then $\Gamma(P_5^*) = \Gamma_4 + \Gamma_5$.
   This case is analogous to the previous one.
3. If there is a vertex $v \in V(H_2)$ colored k, then

$$\Gamma(P_5^*) = \begin{cases} \Gamma_1 + \Gamma_2 + \Gamma_3 & , \, if \; \Gamma_1 \leq \Gamma_4 \\ \Gamma_1 + \Gamma_2 & , \, if \; \Gamma_1 > \Gamma_4 \; and \; \Gamma_3 \leq s_1 \\ \Gamma_2 + \Gamma_3 + \Gamma_4 & , \, if \; \Gamma_1 > \Gamma_4 \; and \; \Gamma_3 > s_1 \end{cases}$$

where $s_1 = \Gamma_1 - \Gamma_4$.
As before, since $N(v) \subseteq V(H_1) \cup V(H_2) \cup V(H_3)$ and $N(v)$ intersects all the stables sets $S_1, \ldots, S_{k-1}$, we have that $\Gamma(P_5^*)$ colors occur in $H_1 \cup H_2 \cup H_3$. Therefore, by Proposition 1, $k = \Gamma(P_5^*) \leq \Gamma_1 + \Gamma_2 + \Gamma_3$.
If $\Gamma_4 \geq \Gamma_1$, then we claim that $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3$. Observe that there are no edges between $V(H_1)$ and $V(H_4)$ and all the edges between $V(H_3)$ and $V(H_4)$. Therefore, an ordering over the vertices of P_5^* that starts by θ_4, θ_1, θ_3 and θ_2, consecutively in this order, produces a greedy coloring of P_5^* with at least $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors, since the colors used by the greedy algorithm to color H_4 are reused to color H_1, and all the colors occurring in H_3 have to be different from the colors occurring in H_4, and hence, in H_1. The result follows.
Otherwise, if $\Gamma_4 < \Gamma_1$, let $s_1 = \Gamma_1 - \Gamma_4$. We study two subcases. At first, if $\Gamma_3 \leq s_1$, then we prove that $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2$. In order to prove this, consider an ordering over $V(P_5^*)$ that starts by θ_1, θ_4, θ_3 and θ_2, consecutively in this order. We claim the greedy algorithm over this ordering uses at least $\Gamma_1 + \Gamma_2$ colors. Indeed, since there are no edges between H_1 and H_4, clearly Γ_4 colors occurring in H_1 will be reused to color H_4. The other s_1 colors in H_1, more precisely Γ_3 out of them, will be sufficient to color H_3, and a total of Γ_1 colors will have been used thus far. Since all the edges between H_1 and H_2 belong to our P_5^*, another Γ_2 previously unused colors will be necessary to color H_2. We now claim that there is no greedy coloring with more than $\Gamma_1 + \Gamma_2$ colors under these hypothesis. Suppose, by contradiction, that there exists an ordering that makes the greedy algorithm generate a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of P_5^* with $p > \Gamma_1 + \Gamma_2$ colors. By Proposition 1 and by the remark that all the colors occur in $H_1 \cup H_2 \cup H_3$, there exists at least one color i that occurs in H_3 and does not occur in H_1 and in H_2.
Recall that, by hypothesis, $\Gamma_3 + \Gamma_4 \leq \Gamma_1$, i.e., $\mathcal{S}'$ has at least $\Gamma_2 + \Gamma_3 + \Gamma_4 + 1$ colors. Since all the colors of $\mathcal{S}'$ occur in $H_1 \cup H_2 \cup H_3$ and $\Gamma_4 < \Gamma_1$, there exists at least one color j that occurs in H_1 and does not occur in $H_2 \cup H_3 \cup H_4$. This is a contradiction, because the vertices of S'_i in H_3 have no neighbor colored j and the vertices of S'_j in H_1 have no neighbor colored i.
Now suppose that $\Gamma_3 > s_1$. We claim that $\Gamma(P_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$. Intuitively, if the colors of H_1 not used in H_4 are not enough to color H_3, then all the s_1 colors of H_1 are used in H_3. Consider an ordering over $V(P_5^*)$ that starts by θ_1, θ_4, θ_3 and θ_2, consecutively in this order. Since there is no edge between $V(H_1)$ and $V(H_4)$, then all, but s_1, colors occurring in H_1 will be reused to color H_4. All these s_1 colors will be necessarily used to partially color H_3. To complete the coloring of H_3, at least $\Gamma_3 - s_1$ new colors will be used. Since there all the edges between $V(H_1)$ and $V(H_2)$, this order leads the greedy algorithm to the generation of a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$ colors.
To prove that $\Gamma(P_5^*) \leq \Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$, we use the same idea as in the previous case. Suppose, by contradiction, that there exists a greedy

coloring $\mathcal{S}'$ of P_5^* with more than $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_1$ colors. Observe that there exist at least $\Gamma_3 - s_1 + 1$ colors that occur in H_3 and do not occur in $H_1 \cup H_2$. Let i be one of these colors. By hypothesis, $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_1 + 1 = \Gamma_2 + \Gamma_3 + \Gamma_4 + 1$ colors. Then, there is a color j that occurs in H_1 and does not occur in $H_2 \cup H_3 \cup H_4$. The existence of colors i and j leads to a contradiction by the same argument used in the preceding case.

4. If there is a vertex $v \in V(H_4)$ colored k, then

$$\Gamma(P_5^*) = \begin{cases} \Gamma_5 + \Gamma_4 + \Gamma_3 & , \textit{if } \Gamma_5 \leq \Gamma_2 \\ \Gamma_5 + \Gamma_4 & , \textit{if } \Gamma_5 > \Gamma_2 \textit{ and } \Gamma_3 \leq s_5 \\ \Gamma_2 + \Gamma_3 + \Gamma_4 & , \textit{if } \Gamma_5 > \Gamma_2 \textit{ and } \Gamma_3 > s_5 \end{cases}$$

   where $s_5 = \Gamma_5 - \Gamma_2$.
   The proof of this case is analogous to the previous one.
5. If there is a vertex $v \in V(H_3)$ colored k, then

$$\Gamma(P_5^*) = \begin{cases} \Gamma_2 + \Gamma_3 + \Gamma_4 & , \textit{if } \Gamma_1 \geq \Gamma_4 \textit{ or } \Gamma_5 \geq \Gamma_2 \\ \Gamma_1 + \Gamma_2 + \Gamma_3 & , \textit{if } \Gamma_1 < \Gamma_4,\ \Gamma_5 < \Gamma_2 \textit{ and } \Gamma_2 - s_4 \geq \Gamma_5 \\ \Gamma_3 + \Gamma_4 + \Gamma_5 & , \textit{if } \Gamma_1 < \Gamma_4,\ \Gamma_5 < \Gamma_2 \textit{ and } \Gamma_2 - s_4 < \Gamma_5 \end{cases}$$

   where $s_4 = \Gamma_4 - \Gamma_1$.
   Again, by Proposition 1 and the fact that there is a vertex colored $k \in V(H_3)$, we have that $\Gamma(P_5^*) \leq \Gamma_2 + \Gamma_3 + \Gamma_4$.
   Suppose first that $\Gamma_1 \geq \Gamma_4$ or $\Gamma_5 \geq \Gamma_2$. We will prove that $\Gamma(P_5^*) = \Gamma_2 + \Gamma_3 + \Gamma_4$. In the case $\Gamma_1 \geq \Gamma_4$, consider any ordering that starts by θ_1, θ_4, θ_2 and θ_3, in this sequence. Alternatively, if $\Gamma_5 \geq \Gamma_2$, consider any ordering that starts by θ_5, θ_2, θ_4 and θ_3, in this sequence. In both cases, these orderings produce a greedy coloring of P_5^* with at least $\Gamma_2 + \Gamma_3 + \Gamma_4$ colors and the proposition follows.
   Now, we define $s_2 = \Gamma_2 - \Gamma_5$. Assume first that $\Gamma_1 < \Gamma_4$ and $\Gamma_5 < \Gamma_2$. Since $\Gamma_1 < \Gamma_4$, an ordering that starts by θ_1, θ_4, θ_2 and θ_3, makes the greedy algorithm generate a coloring with at least $\Gamma_4 + \Gamma_3 + \Gamma_2 - s_4$ colors. Using the hypothesis that $\Gamma_5 < \Gamma_2$, an ordering that starts by θ_5, θ_2, θ_4 and θ_3, consecutively in this order, leads the greedy algorithm to the generation of a greedy coloring with at least $\Gamma_2 + \Gamma_3 + \Gamma_4 - s_2$ colors.
   Now, we need to prove, case by case, that these bounds are also upper bounds. Consider first that $\Gamma_2 - s_4 \geq \Gamma_5$. We claim that $\Gamma(P_5^*) = \Gamma_2 + \Gamma_3 + \Gamma_4 - s_4$. To prove this equality we need only to verify that $\Gamma(P_5^*) \leq \Gamma_4 + \Gamma_3 + \Gamma_2 - s_4$. Suppose, by contradiction, that there is a greedy coloring $\mathcal{S}'$ of P_5^* with more than $\Gamma_4 + \Gamma_3 + \Gamma_2 - s_4$ colors. By Proposition 1 and by hypothesis that $v \in V(H_3)$, there are at least $\Gamma_2 - s_4 + 1$ colors that occur in H_2 and do not occur in $H_3 \cup H_4$. Since, by hypothesis, $\Gamma_5 < \Gamma_2 - s_4 + 1$, there is at least one color i in H_2 that does not occur in $H_3 \cup H_4 \cup H_5$. On the other hand, $\Gamma_2 + \Gamma_3 + \Gamma_4 - s_4 + 1 = \Gamma_1 + \Gamma_2 + \Gamma_3 + 1$, i.e., there is a color j in H_4 that does not occur in $H_1 \cup H_2 \cup H_3$. This is a contradiction,

because neither the vertices of S_i in H_2 have a neighbor colored j nor the vertices of S_j in H_4 have a neighbor colored i.
Finally, suppose that $\Gamma_2 - s_4 < \Gamma_5$. We will prove that $\Gamma(P_5^*) = \Gamma_2 + \Gamma_3 + \Gamma_4 - s_2$. To do this, we use again the symmetry of P_5^*. In the analysis of the previous case, we considered the hypothesis of using the colors of H_4 that do not appear in H_1 to color H_2 and we concluded that if the number of colors of H_2 that do not occur in H_4 is at least Γ_5, we know how to determine the Grundy number of P_5^*.
Using the same idea, we can analogously conclude the following fact: if $\Gamma_4 - s_2 \geq \Gamma_1$, then $\Gamma(P_5^*) = \Gamma_4 + \Gamma_3 + \Gamma_2 - s_2$. Under this hypothesis, using the symmetry, we find the result we needed. However, we can easily verify that $\Gamma_4 - s_2 \geq \Gamma_1$ if, and only if, $\Gamma_2 - s_4 < \Gamma_5$, the proof of this complementary case is analogous to the previous case.

By hypothesis, we know the values of $\Gamma_1, \ldots, \Gamma_5$. Then, the value of $\Gamma(P_5^*)$ can be determined by outputting the maximum value found between among all the cases above. Since we have a constant number of cases, the value of $\Gamma(P_5^*)$ can be found in constant time. Observe that since all the possibilities to place a vertex with the greatest color were checked, $\Gamma(P_5^*)$ is correctly computed. □

Lemma 2. *Given the Grundy numbers of $H_1, \ldots, H_5$, the Grundy number of $\bar{P}_5^* = (H_1 \cup \ldots \cup H_5, E)$ can be determined in constant time.*

Proof: Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of $\bar{P}_5^*$ with $\Gamma(\bar{P}_5^*)$ colors. Analogously to Lemma 1, let us check all the possible cases:

1. There is a vertex $v \in H_1$ colored k, then $\Gamma(\bar{P}_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3$.
   This case can be easily solved because any ordering over $V(\bar{P}_5^*)$ that contains suborderings θ_1, θ_2 and θ_3 produces a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors, since all the colors used in $H_1 \cup H_2 \cup H_3$ must be distinct. Moreover, $\Gamma_1 + \Gamma_2 + \Gamma_3$ is also an upper bound because of Proposition 1 and the hypothesis that $v \in V(H_1)$.
2. If there is a vertex $v \in H_2$ colored k, then:

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_1 + \Gamma_2 + \Gamma_3 & , \textit{if } \Gamma_4 \leq \Gamma_3 \\ \Gamma_1 + \Gamma_2 + \Gamma_4 & , \textit{if } \Gamma_4 > \Gamma_3 \textit{ and } \Gamma_1 \leq \Gamma_5 \\ \Gamma_2 + \Gamma_4 + \Gamma_5 & , \textit{if } \Gamma_4 > \Gamma_3,\ \Gamma_1 > \Gamma_5 \textit{ and } \Gamma_4 - s_1 \geq \Gamma_3 \\ \Gamma_1 + \Gamma_2 + \Gamma_3 & , \textit{if } \Gamma_4 > \Gamma_3,\ \Gamma_1 > \Gamma_5 \textit{ and } \Gamma_4 - s_1 < \Gamma_3 \end{cases}$$

   where $s_1 = \Gamma_1 - \Gamma_5$.
   Consider first that $\Gamma_4 \leq \Gamma_3$. We will prove that $\Gamma(\bar{P}_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3$. Observe that $\Gamma(\bar{P}_5^*) \geq \Gamma_1 + \Gamma_2 + \Gamma_3$, because of an ordering over $V(\bar{P}_5^*)$ that starts by θ_1, θ_2 and θ_3 leads the greedy algorithm to the generation of a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors.
   On the other hand, suppose, by contradiction, that there exists a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of $\bar{P}_5^*$ with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_3 + 1$ colors. As a consequence of Proposition 1, there is a color i such that $S'_i \subseteq V(H_4)$.

Since $\Gamma_4 \leq \Gamma_3$, we conclude that $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 + 1$ colors. Thus, there is a color j such that $S'_j \subseteq V(H_3)$. Consequently, there is no vertex of H_4 colored i adjacent to some vertex of H_3 colored j, i.e., there is no vertex of S'_i adjacent to some vertex of S'_j. This is a contradiction because $\mathcal{S}'$ is a greedy coloring.

Therefore, we can assume that $\Gamma_4 > \Gamma_3$ and set $s_4 = \Gamma_4 - \Gamma_3$. We study two subcases. At first, if $\Gamma_5 \geq \Gamma_1$, then we claim that $\Gamma(\bar{P}^*_5) = \Gamma_1 + \Gamma_2 + \Gamma_4$. Using the hypothesis that $\Gamma_5 \geq \Gamma_1$, we can easily conclude that $\Gamma(\bar{P}^*_5) \geq \Gamma_1 + \Gamma_2 + \Gamma_4$, because an ordering over the vertices of $\bar{P}^*_5$ starting by θ_5, θ_1, θ_4 and θ_2, consecutively in this order, makes the greedy algorithm generate a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_4$ colors.

To show that this value is also an upper bound, suppose, by contradiction, that $\bar{P}^*_5$ admits a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_4 + 1$ colors. By Proposition 1 and by the hypothesis that $v \in V(H_2)$, there is a color i such that $S'_i \subseteq V(H_3)$ (observe that $S'_i \cap V(H_3) \neq \emptyset$ implies that $S'_i \cap V(H_5) = \emptyset$). Since $\Gamma_4 > \Gamma_3$, $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_3 + 2$ colors. Thus, there are at least two colors S'_j and S'_l such that $S'_j \cup S'_l \subseteq V(H_4)$. This contradicts the hypothesis that $\mathcal{S}'$ is a greedy coloring, because neither S'_j nor S'_l has a vertex with some neighbor colored i.

As a consequence, we can suppose that $\Gamma_5 < \Gamma_1$, and if $\Gamma_4 - s_1 \geq \Gamma_3$, then we will prove that $\Gamma(\bar{P}^*_5) = \Gamma_1 + \Gamma_2 + \Gamma_4 - s_1$. Using the hypothesis that $\Gamma_4 > \Gamma_3$ and $\Gamma_5 < \Gamma_1$, we can easily check that an ordering over $V(\bar{P}^*_5)$ starting by θ_1, θ_5, θ_4 and θ_2, consecutively in this order, produces a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_1$ colors.

Suppose, by contradiction, there exists a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of $\bar{P}^*_5$ with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_4 - s_1 + 1$ colors. Since $v \in V(H_2)$, we use Proposition 1 to verify that there are at least $\Gamma_4 - s_1 + 1$ colors that occur only in $H_3 \cup H_4$. Since, by hypothesis, $\Gamma_4 - s_1 \geq \Gamma_3$, there is at least one color i from these $\Gamma_4 - s_1 + 1$ colors that occurs only in H_4. Moreover, since $s_1 = \Gamma_1 - \Gamma_5$, $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_1 + 1 = \Gamma_2 + \Gamma_4 + \Gamma_5 + 1$ colors. Again, the hypothesis that $v \in V(H_2)$ and Proposition 1 imply that there is at least one color j that only occur in $H_1 \cup H_3$. This contradicts the hypothesis that $\mathcal{S}'$ is a greedy coloring because of there are no edges from S'_i to S'_j.

The last case is when $\Gamma_5 < \Gamma_1$ and $\Gamma_4 - s_1 < \Gamma_3$. In this case, $\Gamma(\bar{P}^*_5) = \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$. In order to prove this, observe that $\Gamma_4 - s_1 < \Gamma_3$ if, and only if, $\Gamma_1 - s_4 > \Gamma_5$. Therefore, in order to simplify the proof of this case, we will prove that if $\Gamma_1 - s_4 > \Gamma_5$, then $\Gamma(\bar{P}^*_5) = \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$. To see that $\Gamma(\bar{P}^*_5) \geq \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$, observe that an ordering over $V(\bar{P}^*_5)$ started by θ_4, θ_3, θ_1 and θ_2, consecutively in this order, makes the greedy algorithm generate a greedy coloring with at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_4$ colors.

Suppose, by contradiction, that there is a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ to $\bar{P}^*_5$ with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_4 - s_4 + 1$ colors. Once $v \in V(H_2)$ and the Proposition 1 holds, there are at least $\Gamma_4 - s_4 + 1$ colors occurring only in $H_1 \cup H_3 \cup H_5$. Since $\Gamma_1 - s_4 > \Gamma_5$, there is at least one color i exclusive to $H_1 \cup H_3$. Recall that $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 - s_4 + 1 = \Gamma_1 + \Gamma_2 + \Gamma_3 + 1$

colors. Then, since $v \in V(H_2)$ and by Proposition 1, there exists a color j such that $S'_j \subseteq V(H_4)$. This is a contradiction because of the same previous arguments.

3. If there is a vertex $v \in V(H_3)$ colored k, then

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_1 + \Gamma_3 + \Gamma_2 & , \textit{if } \Gamma_5 \leq \Gamma_2 \\ \Gamma_1 + \Gamma_3 + \Gamma_5 & , \textit{if } \Gamma_5 > \Gamma_2 \textit{ and } \Gamma_1 \leq \Gamma_4 \\ \Gamma_3 + \Gamma_4 + \Gamma_5 & , \textit{if } \Gamma_5 > \Gamma_2,\ \Gamma_1 > \Gamma_4 \textit{ and } \Gamma_5 - s_1 \geq \Gamma_2 \\ \Gamma_1 + \Gamma_2 + \Gamma_3 & , \textit{if } \Gamma_5 > \Gamma_2,\ \Gamma_1 > \Gamma_4 \textit{ and } \Gamma_5 - s_1 < \Gamma_2 \end{cases}$$

where $s_1 = \Gamma_1 - \Gamma_4$.
The proof of this case is analogous to the previous one, taking $s_5 = \Gamma_5 - \Gamma_2$ to play the role of s_4.

4. If there is a vertex $v \in V(H_4)$ colored k, then

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_2 + \Gamma_4 + \Gamma_5 & , \textit{if } \Gamma_1 \geq \Gamma_5 \\ \Gamma_4 + \Gamma_5 & , \textit{if } \Gamma_1 < \Gamma_5 \textit{ and } s_5 \geq \Gamma_2 \\ \Gamma_1 + \Gamma_2 + \Gamma_4 & , \textit{if } \Gamma_1 < \Gamma_5 \textit{ and } s_5 < \Gamma_2 \end{cases}$$

where $s_5 = \Gamma_5 - \Gamma_1$.
Again, observe that, by Proposition 1, the Grundy number in this case is bounded by $\Gamma_2 + \Gamma_4 + \Gamma_5$.
First, suppose that $\Gamma_1 \geq \Gamma_5$. Let us prove that $\Gamma(\bar{P}_5^*) = \Gamma_2 + \Gamma_4 + \Gamma_5$.
In this case, notice that an ordering over $V(\bar{P}_5^*)$ started by θ_1, θ_5, θ_2 and θ_4 leads the greedy algorithm to the generation of a greedy coloring of $\bar{P}_5^*$ with $\Gamma_2 + \Gamma_4 + \Gamma_5$ colors.
Now, assume that $\Gamma_1 < \Gamma_5$. We have to study two cases. In the first case, consider that $s_5 \geq \Gamma_2$. Then, we claim that $\Gamma(\bar{P}_5^*) = \Gamma_4 + \Gamma_5$. To prove this fact, observe that the same ordering over $V(\bar{P}_5^*)$ of the previous case produces a greedy coloring with at least $\Gamma_4 + \Gamma_5$ colors.
In order to show that this is also an upper bound, suppose, by contradiction, that there exists a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of $\bar{P}_5^*$ with $p \geq \Gamma_4 + \Gamma_5 + 1$ colors. Since $v \in V(H_4)$ and Proposition 1 holds, there is a color i that occurs in H_2 and does not occur in $H_4 \cup H_5$. Now, the hypothesis that $s_5 \geq \Gamma_2$ implies that $\mathcal{S}'$ has at least $\Gamma_1 + \Gamma_2 + \Gamma_4 + 1$ colors. As a consequence, there are at least $\Gamma_1 + 1$ colors that occur in H_5 and that do not occur in $H_2 \cup H_4$. By Proposition 1, there is at least one color j from these $\Gamma_1 + 1$ colors such that $S'_j \subseteq V(H_5)$. The fact that there are no edges between S'_i and S'_j contradicts the assumption that $\mathcal{S}'$ is a greedy coloring.
In the complementary case, we have that $\Gamma_1 < \Gamma_5$ and $s_5 < \Gamma_2$. We have to prove now that $\Gamma(\bar{P}_5^*) = \Gamma_4 + \Gamma_5 + \Gamma_2 - s_5$. Observe that the same ordering of the previous case, together with these hypothesis, leads the greedy algorithm to the generation of a greedy coloring of $\bar{P}_5^*$ with at least $\Gamma_4 + \Gamma_5 + \Gamma_2 - s_5$ colors. To verify that this is an upper bound, suppose, by contradiction, that there is a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ with $p \geq \Gamma_4 + \Gamma_5 + \Gamma_2 - s_5 + 1$ colors. The hypothesis that $v \in V(H_4)$ and the

Proposition 1 imply that there are at least $\Gamma_2 - s_5 + 1$ colors exclusive to H_2. Assume that i is one of these colors. Since $\Gamma_4 + \Gamma_5 + \Gamma_2 - s_5 + 1 = \Gamma_4 + \Gamma_2 + \Gamma_1 + 1$, there is also a color j exclusive to H_5. Again, the fact that there are no edges between S'_i and S'_j contradicts the assumption that $\mathcal{S}'$ is a greedy coloring.

5. If there is a vertex $v \in H_5$ colored k, then

$$\Gamma(\bar{P}_5^*) = \begin{cases} \Gamma_3 + \Gamma_5 + \Gamma_4 & , \textit{if } \Gamma_1 \geq \Gamma_4 \\ \Gamma_5 + \Gamma_4 & , \textit{if } \Gamma_1 < \Gamma_4 \textit{ and } s_4 \geq \Gamma_3 \\ \Gamma_1 + \Gamma_3 + \Gamma_5 & , \textit{if } \Gamma_1 < \Gamma_4 \textit{ and } s_4 < \Gamma_3 \end{cases}$$

where $s_4 = \Gamma_4 - \Gamma_1$.
The proof of this case is analogous to the previous one.

Since there is a fixed number of cases to be checked and the calculus to be made in each of them can be also done in constant time, the Grundy number of $\bar{P}_5^*$, given $\Gamma_1, \ldots, \Gamma_5$, can be determined in constant time. $\square$

Lemma 3. *Given the Grundy numbers of $H_1, \ldots, H_5$, the Grundy number of $C_5^* = (H_1 \cup \ldots \cup H_5, E)$ can be determined in constant time.*

Proof: Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of C_5^* with $\Gamma(C_5^*)$ colors. It is enough to prove the Lemma for the case where there is a vertex $v \in V(H_1)$ colored k, since all the other cases follow by symmetry. Therefore, suppose that this is the case. Then:

$$\Gamma(C_5^*) = \begin{cases} \Gamma_1 + \Gamma_2 + \Gamma_3 & , \textit{if } \Gamma_5 \geq \Gamma_2 \textit{ or } \Gamma_4 \geq \Gamma_3 \\ \Gamma_1 + \Gamma_2 + \Gamma_4 & , \textit{if } \Gamma_5 < \Gamma_2,\ \Gamma_4 < \Gamma_3 \textit{ and } \Gamma_2 - s_3 \geq \Gamma_5 \\ \Gamma_1 + \Gamma_3 + \Gamma_5 & , \textit{if } \Gamma_5 < \Gamma_2,\ \Gamma_4 < \Gamma_3 \textit{ and } \Gamma_2 - s_3 < \Gamma_5 \end{cases}$$

where $s_3 = \Gamma_3 - \Gamma_4$.

By Proposition 1 and the hypothesis that $v \in V(H_1)$, $\Gamma(C_5^*) \leq \Gamma_1 + \Gamma_2 + \Gamma_3$.

Assume first that $\Gamma_5 \geq \Gamma_2$ *or* $\Gamma_4 \geq \Gamma_3$. We claim that $\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3$. To prove this, observe that if $\Gamma_5 \geq \Gamma_2$, an ordering over $V(C_5^*)$ that starts by θ_5, θ_2, θ_3 and θ_1, consecutively in this order, makes the greedy algorithm generate a greedy coloring with exactly $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors and the upper bound is achieved. On the other hand, if $\Gamma_4 \geq \Gamma_3$, an ordering over $V(C_5^*)$ that starts by θ_4, θ_3, θ_2 and θ_1, consecutively in this order, produces a greedy algorithm coloring of C_5^* with $\Gamma_1 + \Gamma_2 + \Gamma_3$ colors and, again, the upper bound is achieved.

As a consequence, we can assume that $\Gamma_5 < \Gamma_2$ *and* $\Gamma_4 < \Gamma_3$. Let us set $s_2 = \Gamma_2 - \Gamma_5$ and consider the following subcases. At first, if $\Gamma_2 - s_3 \geq \Gamma_5$, then we prove that$\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_3$. Observe that an ordering over $V(C_5^*)$ started by θ_3, θ_4, θ_2 and θ_1, consecutively in this order, makes the greedy algorithm generate a greedy coloring of C_5^* having at least $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_3$ colors.

Suppose by contradiction that there is a greedy coloring $\mathcal{S}' = \{S'_1, \ldots, S'_p\}$ of C_5^* with $p \geq \Gamma_1 + \Gamma_2 + \Gamma_3 - s_3 + 1$ colors. By the hypothesis that $v \in V(H_1)$

and Proposition 1, there are at least $\Gamma_2 - s_3 + 1$ colors that occur in H_2 and do not occur in $H_1 \cup H_3$. One of them, let us say i, does not occur in H_5, since $\Gamma_2 - s_3 \geq \Gamma_5$. Moreover, as $\Gamma_1 + \Gamma_2 + \Gamma_3 - s_3 + 1 = \Gamma_1 + \Gamma_2 + \Gamma_4 + 1$, we observe that at least $\Gamma_4 + 1$ colors occur in H_3 and that do not occur in $H_1 \cup H_2$. Among them, at least one, j also does not occur in H_4. These facts contradict the assumption that $\mathcal{S}'$ is a greedy coloring, since there are no edges between S'_i and S'_j.

As the last subcase, suppose that $\Gamma_2 - s_3 < \Gamma_5$. We claim that $\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_2$. Notice that $\Gamma_2 - s_3 < \Gamma_5$ if, and only if, $\Gamma_3 - s_2 > \Gamma_4$ and, since if $\Gamma_3 - s_2 > \Gamma_4$, then $\Gamma_3 - s_2 \geq \Gamma_4$. Therefore, the proof of this case is similar to the proof of the previous one up to symmetry, because we can analogously prove that if $\Gamma_3 - s_2 \geq \Gamma_4$, then $\Gamma(C_5^*) = \Gamma_1 + \Gamma_2 + \Gamma_3 - s_2$.

Again, since there is a fixed number of cases to be checked and the calculus to be made in each of them can be also done in constant time, the Grundy number of C_5^* given $\Gamma_1, \ldots, \Gamma_5$ can be determined in constant time. $\square$

In what follows, the two remaining possible types of neighborhood nodes are treated. Recall that G is a fat-extended P_4-laden graph and that $T(G)$ corresponds to its modular decomposition tree.

Lemma 4. *Let v be a neighborhood node of $T(G)$ such that $G(v)$ is isomorphic to a split graph $H = (S'(H) \cup K'(H) \cup R'(H), E)$. Given $\Gamma(G'[R])$, then the Grundy number of $G[M(v)]$ can be determined in linear time.*

Proof: At first, recall that the partition of the vertices of H into sets $S'(H)$, $K'(H)$ and $R'(H)$ can be found in $\mathcal{O}(V(H))$ [15]. Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of $G[M(v)]$ with the maximum number of colors.

Since the strong modules represented by the vertices of $S'(H)$ and $K'(H)$ are stable sets and cliques, respectively, we denote by $S^*(H)$ ($K^*(H)$) the subgraph of $G[M(v)]$ induced by the union of all the modules represented by the vertices of $S'(H)$ (resp., $K'(H)$). Observe that the subgraph of $G[M(v)]$ induced by $V(S^*(H)) \cup V(K^*(H))$ is a split graph and the vertices of $R'(H)$ are adjacent to all the vertices of $K^*(H)$ and to none of $S^*(H)$.

Notice that for any ordering θ over $M(v)$, the greedy algorithm would never assign distinct colors i and j to the vertices of $S^*(H)$, such that $S_i \cup S_j \subseteq S^*(H)$, since $S^*(H)$ is a stable set and so no vertex of S_i would be adjacent to some vertex of S_j. As there is at most one color exclusive to $S^*(H)$, if $R'(H)$ is empty, then $\Gamma(M(v)) \leq |K^*(H)| + 1$. Moreover, an ordering over $V(M(v))$ such that all the vertices of $S^*(H)$ appear before the ones of $K^*(H)$ produces a greedy coloring with $|K^*(H)| + 1$ colors, because of $K'(H)$ is exactly the neighborhood of $S'(H)$.

On the other hand, if $R'(H)$ is not empty, then any greedy coloring of $G[M(v)]$, in particular, $\mathcal{S}$, should assign distinct colors to the vertices of $R'(H)$ and $K^*(H)$, because there are all the edges between the vertices of both sets. Let j be any color occurring in $R'(H)$. If there is a color i such that $S_i \subseteq S^*(H)$, no vertex of S_i would have a neighbor in S_j, contradicting the assumption that $\mathcal{S}$ is a greedy coloring. Consequently, $\Gamma(M(v)) = |K^*(H)| + \Gamma(R'(H))$. As a consequence, $\Gamma(M(v))$ can be computed in linear time following the equation:

$$\Gamma(M(v)) = \begin{cases} |K^*(H)| + \Gamma(R'(H)) & \text{, if } R'(H) \neq \emptyset \\ |K^*(H)| + 1 & \text{, otherwise.} \end{cases}$$

$\square$

Lemma 5. *Let v be a neighborhood node of $T(G)$ such that $G(v)$ isomorphic to a spider $H = (S \cup K \cup R, E)$, f_r be its child corresponding to R, f_2 be its child corresponding to the module which has eventually two vertices and $\Gamma(R)$ be the Grundy number of $G[M(f_r)]$. Then $\Gamma(G[M(v)])$ can be determined in linear time.*

Proof: Suppose that $\mathcal{S} = (S_1, \ldots, S_k)$ is a greedy coloring of $G[M(v)]$ with $\Gamma(G[M(v)])$ colors. If f_2 is trivial, or if f_2 belongs to S and its vertices are not adjacent, or if f_2 belongs to K and its vertices are adjacent, then the Grundy number of $G[M(v)]$ can be found by using the same arguments of Lemma 4, by replacing S, K and R by $S'(G)$, $K'(G)$ and $R'(G)$, respectively.

For otherwise, let x and w be the vertices of f_2. Again, we denote by S^* (K^*) the subgraph of $G[M(v)]$ induced by the union of all the modules represented by the vertices of S (resp., K). We have to check the following cases:

- f_2 belongs to S and x and w are adjacent.

  We claim that for any greedy coloring of $G[M(v)]$, in particular for $\mathcal{S}$, there are no two distinct colors i and j such that $S_i \cup S_j \subseteq S^*$. To show this fact, suppose the contrary. By similar arguments to those used in the proof of Lemma 4, colors i and j must be assigned to x and w. Without loss of generality, suppose that $x \in S_i$ and $w \in S_j$. Since x and w belong to a same module and because of the definition of a spider, there is at least a vertex $y \in K^*$ which is adjacent to none of x and w. Let us suppose that $y \in S_l$. Observe that $(K^* \cup R) \cap S_l = \{y\}$. Now, let u be any other vertex of S^*. So, u has to be assigned to either a color of a non-neighbor in $K \cup R$ or to the smallest between i and j, say i. These facts imply that there is only one vertex of S^*, which is w, colored j and so $(S^* \cup K^*) \cap S_j = \{w\}$. As a consequence, none of w and y has a neighbor colored l and j, respectively. This contradicts the fact that $\mathcal{S}$ is a greedy coloring.

  Therefore, any greedy coloring of $G[M(v)]$ has at most one color containing only vertices of S^*, and then its Grundy number can be determined in linear time by using similar arguments to those used in Lemma 4.

- f_2 belongs to K and x and w are not adjacent.

  We claim that there are no distinct colors i and j, such that $x \in S_i$ and $w \in S_j$. For otherwise, since x and w are not adjacent and the belong to the same module, either w would not have a neighbor colored i or x would not have a neighbor colored j. Therefore, by similar arguments to those used in the proof of Lemma 4, we can conclude that the Grundy number of $G[M(v)]$ can be found in linear time.

□

Theorem 2. *If $G = (V, E)$ is a fat-extended P_4-laden graph and $|V| = n$, then $\Gamma(G)$ can be found in $\mathcal{O}(n^3)$.*

Proof: The algorithm computes $\Gamma(G)$ by traversing the modular decomposition tree of G in a post-order way and determining the Grundy of each inner node of $T(G)$ based on the Grundy number of its children. The modular decomposition tree can be found in linear time [14], the post-order traversal can be done in $\mathcal{O}(n^2)$ and the Grundy number of each inner node can be found in linear time, because of Lemmas 1, 2, 3, 4 and 5, and because of the results of Gyárfás and Lehel for cographs [2]. □

Corollary 1. *Let G be a graph that belongs to one of the following classes: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden. Then, $\Gamma(G)$ can be determined in polynomial time.*

Proof: According to the definition of these classes [13], they are all strictly contained in the fat-extended P_4-laden graphs and so the corollary follows. □

5. Conclusions

We extended the previous result that states that the Grundy number can be determined in polynomial time for cographs [2], which are exactly the P_4-free graphs, to a greater class of graphs that we called fat-extended P_4-laden graphs. In fact, by observing that every complement of a bipartite graph is P_5-free, the result of Zaker [5] implies that determining the Grundy number for a P_5-free graph is also NP-hard.

The problems of finding a minimum vertex coloring, a minimum clique cover, a maximum clique and a maximum independent set can be solved in polynomial time for extended P_4-laden graphs [12, 16]. We remark that these results can be easily extended to $f$at-extended P_4-laden graphs. Even thought the vertex coloring problem can be solved in polynomial time for fat-extended P_4-laden graphs, the study of the Grundy number also provide bounds to other problems, like Weighted Coloring, whose complexity is not determined even for a subclass of extended P_4-laden graphs called P_4-sparse graphs [17, 18].

Finally, we observe that since Lemmas 1, 2 and 3 were proved without the assumption that we were dealing with fat-extended P_4-laden graphs, those results can be useful for any class of graphs whose modular decomposition contains fat neighborhood nodes.

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