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Thesis defended by **Francisco LOPEZ RAMIREZ**

Defended on **19th November, 2018**

In order to become Doctor from Université de Lille

Academic Field **Control**

Control and Estimation in Finite-Time and in Fixed-Time via Implicit Lyapunov Functions

Thesis supervised by	Denis EFIMOV	Co-Advisor
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Contrôle et Estimation en Temps Fixe et en Temps Fini via Fonctions de Lyapunov Implicites

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Table of Contents

Acronyms

A | F | G | I | L | M | N | P | S

A

AS Asymptotic Stability/Stable.

F

FT Finite Time.
FTS Finite-Time Stability/Stable.
FxT Fixed-Time.
FxTS Fixed Time Stability/Stable.

G

GAS Global Asymptotic Stability/Stable.

I

ILF Implicit Lyapunov Function.
ISpS Input-to-State Practical/Practially Stability/Stable.
ISS Input-to-State Stability/Stable.

L

LDM Lyapunov's Direct Method.
LF Lyapunov Function.

M

MIMO Multiple-Input Multiple-Output.

N

NonA Nonasymptotic.

P

PDE Partial Differential Equation.

S

SISO Single-Input Single-Output.
SMC Sliding Mode Control.

Notations

$\mathbb{R}$ The set of real numbers.

$\mathbb{R}_+$ The set of strictly positive real numbers: $\{x \in \mathbb{R} : x > 0\}$.

$\mathbb{R}_{\geq 0}$ The set of nonnegative real numbers: $\{x \in \mathbb{R} : x \geq 0\}$.

$\|\cdot\|$ The Euclidean norm.

$|\cdot|$ Absolute value of a real number.

$\lceil\cdot\rfloor^{\nu}$ Short notation of $|\cdot|^{\nu}\mathrm{sign}(\cdot)$.

$\overline{1,n}$ Denotes the sequence $1,\ldots,n$.

$\partial\mathcal{A}$ The boundary of the set $\mathcal{A}$.

$\|x\|_{\mathcal{A}}$ The distance from a point $x \in \mathbb{R}^n$ to a set $\mathcal{A} \subset \mathbb{R}^n$: $\inf_{\xi\in\mathcal{A}} \|x-\xi\|$.

$\partial_x \rho(x,y,\ldots)$ Denotes the partial derivative of $\rho(x,y,\ldots)$ with respect to x.

$\mathcal{B}$ The unitary ball: $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

$\mathcal{B}(s)$ A ball of size s: $\{x \in \mathbb{R}^n, s > 0 : \|x\| \leq s\}$.

$\mathcal{S}$ The unit shpere: $\{x \in \mathbb{R}^n : \|x\| = 1\}$.

$\mathcal{B}_r$ The unitary ball in the homogeneous norm $\|x\|_r$: $\{x \in \mathbb{R}^n : \|x\|_r \leq 1\}$.

$\mathcal{B}_r(s)$ A ball in the homogeneous norm $\|x\|_r$ of size s: $\{x \in \mathbb{R}^n, s > 0 : \|x\| \leq s\}$.

$\mathcal{S}$ The unit shpere in the homogenenous norm $\|x\|_r$: $\{x \in \mathbb{R}^n : \|x\| = 1\}$.

I_n The identity matrix $I \in \mathbb{R}^{n\times n}$.

$\mathrm{diag}(z_1,\ldots,z_n)$ A diagonal matrix with diagonal elements z_i, $i = \overline{1,m}$.

$\mathbb{1}_k$ A column vector of ones of size k.

$\lambda_{\min}(P), \lambda_{\max}(P)$ The minimum and the maximum eigenvalues of the matrix P, resp.

$\mathrm{rown}(P), \mathrm{coln}(P)$ The number of rows and the number of columns of a matrix P, resp.

$\ker(P), \mathrm{range}(P)$ The null space and the column space of the matrix P, resp.

$\mathrm{null}(P)$ The matrix with columns defining the orthonormal basis of the subspace $\ker(W)$.

Résumé Long

Ce manuscrit est composé de 5 chapitres ainsi que d'une conclusion générale et liste bibliographique. La thèse porte sur la propriété de stabilité en temps fini pour les systèmes dynamiques. Une attention particulière est portée au cas de figure où le temps de convergence est une fonction bornée de l'espace d'état. Cette propriété, appelé stabilité en temps fixe, est étudiée dans le cas de l'estimation d'état et du bouclage de sortie stabilisant en systèmes linéaires et non-linéaires.

Une première partie présente une introduction au sujet, donne quelques outils théoriques et situe la contribution de la thèse. Ainsi, le chapitre 1 concerne les pré-requis nécessaires á la lecture des chapitres suivants. Les théorèmes qui conforment la base théorique de la thèse sont introduits de manière détaillé. Dans ce chapitre sont rappelées les notions de stabilité puis est discutée la notion de stabilité en temps fini et enfin en temps fixe. Un certains nombres d'outils qui seront bien utiles par la suite sont introduits. Toutes les notions introduites sont illustrées par des exemples.

Les premiers contributions théoriques se trouvent dans le chapitre 2. Dans ce chapitre est introduit des caractérisations générales de stabilité en temps temps fixe en terme de fonction de Lyapunov. L'idée principale de cette étude est d'utiliser, á l'image de ce qui se fait généralement dans les théorèmes de Lyapunov inverses, le temps de convergence á partir d'un point comme une fonction de Lyapunov.

Deux cas des conditions nécessaires et suffisantes sont introduites, les cas général de la fonction de temps de convergence et le cas où cette fonction est continue. Les résultats sont alors étendus dans le cadre des systèmes avec commande. Ainsi, il est introduite le concept de fonction de Lyapunov de contrôle (CLF en anglais) à temps fixe qui, en ajoutant une hypothèse de *small control property* garanti l'existence d'une loi de commande qui assure la stabilisation en temps fixe.

Dans le chapitre 3, les propriétés de Stabilité Entrée-État (ISS en anglais) dans le cas où le système autonome vérifie la propriété de stabilité en temps fini et fixe est introduite; cette propriété est appelé nonA ISS. En s'inspirant des travaux de Eduardo Sontag, le concept de function de class KL généralisée est aussi introduit. Les théorèmes 3.1 et 3.2 caractérisent en terme de fonctions de Lyapunov ces propriétés ISS en temps fixe et fini. Une hypothèse supplémentaire de régularité sur le temps de convergence est alors introduite pour montrer le caractère nécessaire de l'existence d'une fonction de Lyapunov nonA-ISS. Ce chapitre clôture avec une caractérisation analogue en utilisant de fonctions de Lyapunov implicites.

Les chapitres 4 et 5 concernent la synthèse d'algorithmes d'estimation et de commande exploitant les outils développés précédemment. Le chapitre 4 s'attaque à la synthèse d'observateur pour les systèmes linéaires. Les outils développés jusqu'á présent sont alors exploités dans ce contexte pour donner l'existence d'un observateur qui converge en temps fini et fixe. L'outil central est l'utilisation de la fonction de Lyapunov implicite. Ce type d'outil permet l'introduction d'inégalités matricielles linéaires qui sont un certificat à la possibilité de concevoir un observateur d'état en temps fini et fixe. Dans un premier temps une condition LMI à dimension infinie est présentée (les conditions matricielles doivent être vérifiées pour un continuum de paramètres); en ajoutant des conditions LMI supplémentaires on obtient des conditions de dimension finie tout en permettant de fournir une estimation du temps de convergence.

Dans le chapitre 5, le problème de bouclage de sortie est considéré pour une chaîne d'intégrateurs. Une stratégie de commutation entre deux degrés d'homogénéité permet de synthétiser un observateur et un contrôleur qui convergent en temps fixe . Une condition LMI est introduite permettant l'obtention des paramètres optimales du schema de contrôle et qui permet d'ajuster le temps de convergence.

General Introduction

Lyapunov stability (LS) and asymptotic stability (AS) are properties of dynamical systems that refer to a particular behavior of the system's trajectories with respect to an equilibrium point. Lyapunov stable dynamics allow the system's trajectories to be increasing, decreasing or constant, as long as they remain contained inside a ball proportional to magnitude of deviation of the initial conditions from the equilibrium. Asymptotic stability asks furthermore that the trajectories approach the equilibrium point as time approaches infinity, in other words, AS states that given a sufficiently large amount of time, the trajectories will arrive arbitrarily close to the equilibrium point. In this case, locally increasing or decreasing trajectories are allowed as long as they remain contained in a ball approaching zero.

Exponential stability (ES), for instance, occurs when the evolution of the system's trajectories can be upper bounded by an exponential function, thus revealing more about the system's behavior and its convergence rate. However, as in the AS case, the trajectories of systems with an exponentially stable origin will converge exactly to zero in an infinite amount of time.

A more recently described type of stability, early developed in the works of Zubov [Zubov, 1958], Roxin [Roxin, 1966] and Haimo [Haimo, 1986], involves the existence of an initial-condition dependent, scalar function T that provides the time for which all the system's trajectories converge exactly to the equilibrium. Here, the rate of convergence, instead of being characterized by a trajectory bounding function, is determined by the function T, often called the *settling-time* function. Since T is a real-valued function, this type of stability is called *finite-time stability* (FTS). Fixed-time stability (FxTS) refers to the case when the settling-time function of an FTS system is bounded by a finite value in the whole domain of trajectories' convergence. This property implies that for any initial condition, no matter how far from the origin, all the system's trajectories will converge to the origin before a fixed and *a priori* known time. Another term used to describe this property is uniformity with respect to the initial conditions. To distinguish them from AS, both FTS and FxTS are often referred as *nonasymptotic* (NonA) stability.

The main tool to establish FTS or FxTS of a given system is based on the theory of homogeneous systems. Homogeneity is a symmetry-like property in which a multiplicative scaling of the arguments of a function results in a proportional scaling of the function. Through this multiplicative regularity, stability and robustness analysis can be significantly simplified.

The motivation for studying FTS systems in control theory is quite varied. Whenever control and estimation goals are heavily time-constrained, they provide strict theoretical estimates on the convergence time. Also, since the separation principle is not satisfied in the general nonlinear case, these estimates can be further used to independently design FTS observers and controllers and guarantee the stability of its combined application [Hong et al., 2000]. Along the same lines, FTS can be used to assert stability of interconnected systems [Zoghlami et al., 2013]. In terms of robustness and in comparison with exponentially stable systems, it has been shown that FTS leads to an improved rejection of low-level persistent disturbances [Bhat and Bernstein, 2000]. In sliding mode control theory, FTS constitutes a key feature [Polyakov and Fridman, 2014].

Several other works have shown ingenious implementations of FTS control and estimation algorithms. These works include applications to secure network communications [Perruquetti et al., 2008], finite-time regulation of robot manipulators [Hong et al., 2002], fault detection [Floquet et al., 2004], multi-agent consensus [Wang et al., 2016] and synchronization [Du et al., 2011].

All the types of stability mentioned so far can be studied through Lyapunov analysis, this is, by proposing a continuous positive definite scalar function V such that its derivative along the trajectories of the system satisfies, for each type of stability, a particular differential inequality. Therefore, the four of them benefit from the main advantage of Lyapunov analysis as well as suffer from its main drawback. The former being the ability to determine the stability of a system without linearizing it nor calculating explicitly its trajectories, and the latter the absence of a general procedure to propose Lyapunov functions.

To expand the notions of Lyapunov analysis and to circumvent some its drawbacks, the works of Adamy [Adamy, 2005] and Korobov [Korobov, 1979] propose an alternative stability framework using implicit Lyapunov functions (ILF). The general idea is to propose an implicitly defined function that satisfies certain properties in order to guarantee asymptotic stability without the need of an explicit expression of V nor of its derivative. An extension to the FT and the FxT case has been done in [Polyakov et al., 2015]. Clearly, this approach will also lack a general constructive procedure to find ILFs, however, as will be shown in this work, there exist FTS systems, in particular homogeneous systems, for which no explicit Lyapunov function is known, yet they admit an implicit one. Another relevant case, addressed in Chapter 4, is the nonasymptotic stabilization and observation of linear systems, where once more explicit Lyapunov functions that assert NonA stability are not known. Here, the implicit approach will again prove to be useful.

Regardless of the stability type, robustness against perturbations becomes a central topic in control theory since any real phenomena is exposed to them and they may alter or completely destroy the stability properties of a given system. Robustness is a broad term that refers to the coping capabilities of a given system when internal or external disturbances are present. Input-to-state stability (ISS) is a widespread robustness framework for nonlinear system that addresses the property that for any bounded input, the system's state will remain bounded. In the case of linear systems, any stable system possesses this feature, in nonlinear systems however, it becomes much more difficult to assert it. Once more, this property can be characterized using Lyapunov theory and the concept of ISS Lyapunov functions, developed for the most part by E. Sontag [Sontag and Wang, 1995].

The present work revolves around three main concepts: NonA stability, homogeneous systems and the implicit Lyapunov function approach. In Chapter 1 we present the theoretical framework and the formal definitions of the main concepts. Chapters 2 and 3 focus on theoretical contributions. In the former, necessary and sufficient conditions for NonA stability for continuous autonomous systems are presented. In the latter, a framework that gathers ISS Lyapunov functions, NonA stability analysis and the implicit framework is introduced. By joining these notions, the stability rate and the robustness properties of a given system can be proven with a single, implicitly defined Lyapunov function. Chapters 4 and 5 deal with more practical aspects. In Chapter 4, FT and FxT convergent observers are designed for linear MIMO systems using the implicit approach. In Chapter 5, homogeneity properties and the implicit approach are used to design an FxT output controller for the chain of integrators.

Chapter 1

Theoretical Background

Contents

This chapter presents the formal definitions of the three main concepts on which this work is based, that is, convergence rates, homogeneity and input-to-state stability. As we discuss them, it will become clear the relationship with one another and the goals set for this work.

1.1 Stability Rates in Nonlinear Systems

Throughout this work, we will consider the domain $\Gamma \subset \mathbb{R}^n$ as an open connected set containing the origin and $\partial\Gamma$ will denote its boundary. Consider the autonomous system

$$\dot{x} = f(x), \quad x \in \Gamma, \tag{1.1}$$

where $f : \Gamma \to \mathbb{R}^n$ is a continuous function and $f(0) = 0$. Let us assume that f is such that (1.1) has the properties of existence and uniqueness of solutions in forward time outside the origin. Then $\psi_x(t)$, alternatively $\psi(x,t)$, denotes the solution to system (1.1) starting from $x \in \Gamma$ at $t = 0$.

Definition 1.1 (Bacciotti and Rosier, 2005; Khalil, 2002; Polyakov, 2012). The origin of the system (1.1) is said to be:

Lyapunov stable if for any $x_0 \in \Gamma$ the solution $\psi_{x_0}(t)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there exists some $\delta > 0$ such that for any $x_0 \in \Gamma$, if $\|x_0\| \leq \delta$ then $\|\psi_{x_0}(t)\| \leq \epsilon$ for all $t \geq 0$;

asymptotically stable if it is Lyapunov stable and $\|\psi_{x_0}(t)\| \to 0$ as $t \to +\infty$ for any $x_0 \in \Gamma$;

finite-time stable if it is Lyapunov stable and *finite-time converging* from Γ, *i.e.* for any $x_0 \in \Gamma$ there exists $0 \leq T < +\infty$ such that $\psi_{x_0}(t) = 0$ for all $t \geq T$. The function $T(x_0) = \inf\{T \geq 0 : \psi_{x_0}(t) = 0\ \forall t \geq T\}$ is called the *settling-time* function of system (1.1);

fixed-time stable if it is finite-time stable and $\sup_{x_0 \in \Gamma} T(x_0) < +\infty$.

The set Γ is called the *domain of attraction*. If $\Gamma = \mathbb{R}^n$, then the corresponding properties become *global*.

Stability notions can be similarly defined with respect to a set, by replacing the distance to the origin in Definition 1.1 with the distance to an invariant set. For example, global finite-time stability with respect to a set $\mathcal{A} \subset \mathbb{R}^n$ is equivalent to the following two properties:
i) *Lyapunov stability*: for any $x_0 \in \mathbb{R}^n$ the solution $\psi_{x_0}(t)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there exists some $\delta > 0$ such that if $\|x_0\|_{\mathcal{A}} \leq \delta$ then $\|\psi_{x_0}(t)\|_{\mathcal{A}} \leq \epsilon$ for all $t \geq 0$;
ii) *finite-time stability*: for any $x_0 \in \mathbb{R}^n$ there exists $0 \leq T < +\infty$ such that $\|\psi(t, x_0)\|_{\mathcal{A}} = 0$ for all $t \geq T$.

Example 1.1

The scalar system

$$\dot{x} = -\lceil x \rfloor^{\alpha}, \tag{1.2}$$

where $\alpha \in (0,1)$, has an equilibrium at the origin, is continuous on $\mathbb{R}$ and locally Lipschitz everywhere except at the origin. Hence, every initial condition on $\mathbb{R}\backslash\{0\}$ has a unique solution in forward time. The solutions of (1.2) can be obtained by direct integration as

$$\psi_x(t) = \begin{cases} \mathrm{sign}(x)[|x|^{1-\alpha} - (1-\alpha)t]^{\frac{1}{1-\alpha}}, & t < \frac{1}{1-\alpha}|x|^{1-\alpha} \\ 0, & t \geq \frac{1}{1-\alpha}|x|^{1-\alpha} \end{cases}.$$

Since any trajectory will be exactly 0 for any $t \geq \frac{1}{1-\alpha}|x|^{1-\alpha}$, the settling-time function is given by $T(x) = \frac{1}{1-\alpha}|x|^{1-\alpha}$. Let us compare this with the exponentially stable linear system $\dot{y} = -y$, whose trajectories are given by $\psi_y(t) = ye^{-t}$. It is clear that the trajectories will be exactly zero in an infinite amount of time. Figure 1.1 shows that starting at the same initial condition, $\psi_x(t)$ converges to zero around $t = 2$ while $\psi_y(t)$ continues to decrease exponentially but never reaching 0 exactly.

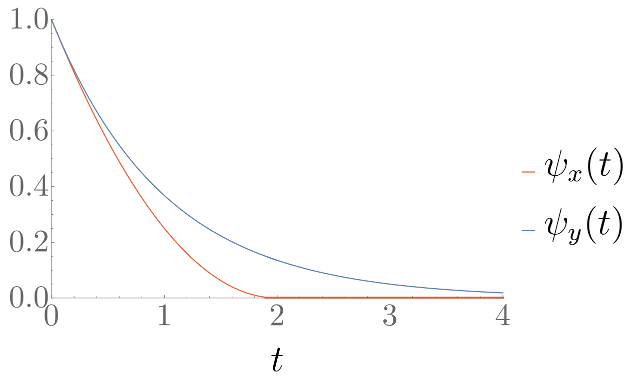

Figure 1.1 – Trajectories $\psi_x(t)$ and $\psi_y(t)$ of the scalar systems $\dot{x} = -\lceil x \rfloor^{\alpha}$ and $\dot{y} = -y$, for $x(0) = y(0) = 1$ and $\alpha = \frac{1}{2}$.

The definition of finite-time stability does not address the regularity of the settling-time function. The following example provides a discontinuous settling-time function at the origin (the function T becomes infinite when approaching the origin from a particular direction).

Example 1.2 (Bhat and Bernstein, 2000).

Consider the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined in Figure 1.2 on the quadrants

$$\mathcal{Q}_I = \{x \in \mathbb{R}^2 \backslash \{0\} : x_1 \geq 0, x_2 \geq 0\}, \quad \mathcal{Q}_{II} = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 \geq 0\},$$
$$\mathcal{Q}_{III} = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 < 0\}, \quad \mathcal{Q}_{IV} = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\},$$

where $f(0) = 0$, $r > 0$, $\theta \in [0, 2\pi)$ and $x = (x_1, x_2) = (r\cos\theta, r\sin\theta)$.
In [Bhat and Bernstein, 2000], it is shown that f is continuous on $\mathbb{R}^2$, Lipschitz everywhere except on the x_1 and x_2 axes, that the vector field (1.1) has unique solutions in forward time and that its origin is globally finite-time stable. However, when approaching the origin from the negative part of the x_2-axis, the settling-time function tends to infinity. To prove this, consider the sequence $\{x_m\}$ where $x_m = (0, -\frac{1}{m}), m = \overline{1, \infty}$. We have that $x_m \to 0$ as $m \to \infty$ and since $\dot{\theta} = -r$ on $\mathcal{Q}_{III}$, the time taken for a solution starting at x_m to enter $\mathcal{Q}_{II}$, before converging to the origin, is equal to $\frac{\pi}{2\sqrt{x_{m_1}^2 + x_{m_2}^2}} = \frac{\pi m}{2}$. Since any solution that starts on $\mathcal{Q}_{III}$ must enter $\mathcal{Q}_{II}$ before converging to the origin, it follows that $T(x_m) \geq \frac{\pi m}{2}$, for each $x_m \in \mathcal{Q}_{III}$, and therefore $T(x_m) \to \infty$ as $x_m \to 0$. In Figure 1.2, the spacing of the arrows shows that for the trajectories starting from the negative part of the x_2 axis, the closer they start from the origin, the longer it will take for them to converge.

Example 1.2 motivates the study of the regularity of the settling-time function. In order to do so, more insight about the solutions of an FTS system is needed. Let us start that by showing that all solutions are well defined.

Proposition 1.1 (Bhat and Bernstein, 2000). *Suppose that the origin of* (1.1) *is finite-time stable. Then, ψ is defined on $\mathbb{R}_{\geq 0} \times \Gamma$ and $\psi(t, x) = 0$ for allt $\geq T(x)$, $x \in \Gamma$, where $T(0) := 0$.*

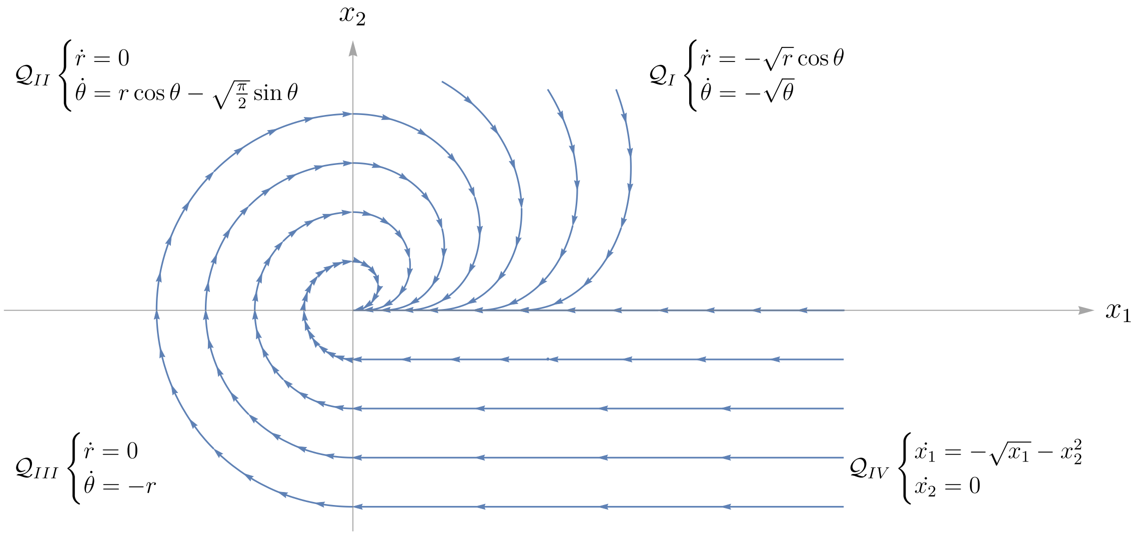

Figure 1.2 – Phase portrait of system defined by quadrants in the picture. The trajectories show the behavior of a global finite-time stable system with discontinuous settling-time function.

Proof. It can be shown that LS of the origin implies that $\psi = 0$ is the unique solution of (1.1), this proves that 0 is contained in the domain of definition of ψ_x, *i.e.* the set Γ, and that $\psi_0 = 0$. Let $y \in \Gamma \backslash \{0\}$ and define

$$x(t) = \begin{cases} \psi(t,y), & 0 \leq t \leq T(y), \\ 0, & t \geq T(y). \end{cases} \tag{1.3}$$

By construction, x is continuously differentiable on $\mathbb{R}_{\geq 0} \backslash \{T(y)\}$ and satisfies (1.1) on $\mathbb{R}_{\geq 0} \backslash \{0\}$. Also, it follows from the assumption of continuity of f that

$$\lim_{t \to T(y)^-} \dot{x}(t) = \lim_{t \to T(y)^-} f(x(t)) = 0 = \lim_{t \to T(y)^+} \dot{x}(t), \tag{1.4}$$

so that x is continuously differentiable at $T(y)$ and satisfies (1.1). Thus $x(t)$ is a solution of (1.1) on $\mathbb{R}_{\geq 0}$. To prove uniqueness, suppose that z is a solution of (1.1) on $\mathbb{R}_{\geq 0}$ satisfying $z(0) = y$. Then by the uniqueness assumption, x and z agree on $[0, T(y))$. By continuity, x and z must also agree on $[0, T(x)]$ so that $z(T(y))$. Lyapunov stability now implies that $z(t) = 0$ for $t > T(y)$. This proves uniqueness. Thus ψ_x is defined and unique on $\mathbb{R}_{\geq 0}$ and satisfies $\psi_x(t) = 0$ on $[T(x), \infty)$ for every $x \in \Gamma$. ■

The next lemma introduces two key properties of the settling-time function that will be extensively used in the chapters that follow.

Lemma 1.1 (Bhat and Bernstein, 2000). *Suppose the origin of* (1.1) *is a finite-time stable equilibrium. Then the following statements hold:*

i) If $x \in \Gamma$ and $t \in \mathbb{R}_{\geq 0}$, then $T(\psi_x(t)) = \max\{T(x) - t, 0\}$.

ii) T is continuous on Γ if and only if T is continuous at 0.

Due to its importance for the topic of this thesis, below we present the proof of this lemma, taken from [Bhat and Bernstein, 2000].

Proof of i). From the definition of T and uniqueness of solutions, we obtain

$$\psi(T(x)+t,x) = 0 \tag{1.5}$$

$$\psi(t,\psi(h,x)) = \psi(t+h,x) \tag{1.6}$$

for all $x \in \Gamma$ and all $t,h \in \mathbb{R}_{\geq 0}$. From (1.5) and (1.6) we obtain the chain of equalities

$$\psi(T(x)+t-t,x) = \psi(T(x)-t,\psi(t,x)) = \psi(T(\psi(t,x)),\psi(t,x)) = 0$$

and from the definition of the settling-time function T we have that

$$T(\psi(t,x)) \leq T(x)-t \quad \forall t \leq T(x). \tag{1.7}$$

On the other hand, we have that

$$\psi(T(x),x) = \psi(T(\psi(t,x)),\psi(t,x)) = \psi(T(\psi(t,x))+t,x) = 0.$$

By the definition of the settling-time function T, $T(x) \leq T(\psi(t,x))+t$, therefore

$$T(\psi(t,x)) \geq T(x)-t \quad \forall t \leq T(x). \tag{1.8}$$

Gathering (1.8) and (1.7), and considering that Proposition 1.1 implies that T can be extended by defining $T(0) = 0$, the statement *i)* follows. ■

Proof of ii). Necessity is immediate. To prove sufficiency, suppose that T is continuous at 0. Let $z \in \Gamma$ and consider a sequence $\{z_m\}$ in Γ that converges to z. Let $\tau^- = \liminf_{m\to\infty} T(z_m)$ and $\tau^+ = \limsup_{m\to\infty} T(z_m)$. Note that both τ^- and τ^+ are in $\overline{\mathbb{R}}_{\geq 0}$ and that

$$\tau^- \leq \tau^+. \tag{1.9}$$

Next, let $\{z_l^+\}$ be a subsequence of $\{z_m\}$ such that $T(z_l^+) \to \tau^+$ as $l \to \infty$. The sequence $\{(T(z),z_l^+)\}$ converges in $\mathbb{R}_{\geq 0} \times \Gamma$ to $(T(z),z)$. Proposition 1.1 implies that if the origin of (1.1) is FTS, then the solutions of (1.1) define a continuous global semiflow on Γ, this is, $\psi : \mathbb{R}_{\geq 0} \times \Gamma \to \Gamma$ is a (jointly) continuous function. Then, by continuity of ψ and the equation (1.5), $\psi(T(z),z_l^+) \to \psi(T(z),z) = 0$ as $l \to \infty$. Since T is assumed to be continuous at 0, $T(\psi(T(z),z_l^+) \to T(0) = 0$ as $l \to \infty$. Using the property i) with $t = T(z)$ and $x = z_l^+$, we obtain $\max\{T(z_l^+) - T(z), 0\} \to 0$ as $l \to \infty$. Thus $\max\{\tau^+ - T(z), 0\} = 0$, that is,

$$\tau^+ \leq T(z). \tag{1.10}$$

Now, let $\{z_l^-\}$ be a subsequence of $\{z_m\}$ such that $T(z_l^-) \to \tau^-$ as $l \to \infty$. It follows from (1.9) and (1.10) that $\tau^- \in \mathbb{R}_{\geq 0}$. Therefore, the sequence $\{(T(z_l^-), z_l^-)\}$ converges in $\mathbb{R}_{\geq 0} \times \Gamma$ to (τ^-, z). Since

ψ is continuous, it follows that $\psi(T(z_l^-), z_l^-) \to \psi(\tau^-, z)$ as $l \to \infty$. Equation (1.5) implies that $\psi(T(z_l^-), z_l^-) = 0$ for each l. Hence $\psi(\tau^-, z) = 0$ and by the definition of T (Definition 1.1),

$$T(z) \leq \tau^-. \tag{1.11}$$

From (1.9), (1.10) and (1.11) we conclude that $\tau^- = \tau^+ = T(z)$ and hence $T(z_m) \to T(z)$ as $m \to \infty$. ■

Lyapunov Analysis

In this subsection, we introduce the main Lyapunov analysis results, stated as sufficient conditions. As it is well known, they provide means to determine the stability of general nonlinear systems without the need of explicitly calculate the solutions of the system. The following definitions and notations will be used: a continuous function $V : \Gamma \to \mathbb{R}_{\geq 0}$ is said to be *radially unbounded* on Γ if $V(x) \to +\infty$ as $\|x\|_{\partial\Gamma} \to 0$. If Γ is unbounded then, in addition, $V(x) \to +\infty$ as $\|x\| \to +\infty$, $x \in \Gamma$.

When dealing with functions that are not everywhere differentiable, the Dini derivatives are often used. They generalize the concept of the derivative by taking into account the limit's side and either the supremum or the infimum of the limit. Thus, the upper-right Dini derivative of a function $g : [a,b) \to \mathbb{R}$, $b > a$, is the function $D^+g : [a,b) \to \overline{\mathbb{R}}$ given by $D^+g(t) = \limsup_{h\to 0^+} \frac{1}{h}[g(t+h) - g(t)]$, $t \in [a,b)$. If g is differentiable at t, then $D^+g(t)$ is the ordinary derivative of g at t. For a continuous function $V : \Gamma \to \mathbb{R}$, the upper-right Dini derivative of V along the solutions of (1.1) is given by

$$\dot{V}(x) = D^+(V \circ \psi_x)(0), \tag{1.12}$$

if V is continuously differentiable on $\Gamma\backslash\{0\}$, then

$$\dot{V}(x) = \frac{d(V \circ \psi_x)}{dt}(0) = \frac{\partial V}{\partial x} f(x), \quad x \in \Gamma\backslash\{0\}. \tag{1.13}$$

A function $V : \Gamma \to \mathbb{R}$ is said to be *proper* if $V^{-1}(K)$ is compact for every compact set $K \subset \mathbb{R}$. Note that if $\Gamma = \mathbb{R}^n$ and V is continuous and radially unbounded, then V is proper [Bhat and Bernstein, 2000].

Theorem 1.1 ***[Lyapunov's Direct Method]*** *(Khalil, 2002; Krasovskij, 1963). Let $x = 0$ be an equilibrium of (1.1). Let $V : \Gamma \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function such that*

i) $V(0) = 0$ *and* $V(x) > 0$ *in* $\Gamma\backslash\{0\}$.

ii) $\dot{V}(x) \leq 0$ *in* Γ.

Then, $x = 0$ is LS. Moreover, if V is radially unbounded and

iii) $\dot{V}(x) < 0$ *in* $\Gamma\backslash\{0\}$,

then $x = 0$ is AS on Γ.

Typically, a function V satisfying *i)* and *ii)* is called a *Lyapunov function* (LF), whereas a function V that satisfies *i)* and *iii)* is called a *strict* LF. However, depending on the context, the regularity of the function V varies. In this work we will adopt the following terminology:

Definition 1.2. A function $V : \Gamma \to \mathbb{R}_{\geq 0}$ is called a *strict Lyapunov function* for system (1.1) if it fulfills the following properties:

i) *Positive Definiteness.* $V(x) > 0$ for all $x \in \Gamma \setminus \{0\}$ and $V(0) = 0$.

ii) V is radially unbounded on Γ.

iii) V is continuously differentiable on Γ and $\dot{V}(x) < 0$ for each $x \in \Gamma\backslash\{0\}$.

As the next theorems show, finite-time and fixed-time stability with continuous settling-time function can also be asserted through Lyapunov analysis.

Theorem 1.2 (Bhat and Bernstein, 2000). *Suppose there exist a positive definite continuous function $V : \Gamma \to \mathbb{R}$ and real numbers $c > 0$ and $\alpha \in (0,1)$ such that*

$$\dot{V}(x) \leq -cV(x)^{\alpha}, \quad \forall x \in \Gamma\backslash\{0\}. \tag{1.14}$$

Then the origin of (1.1) *is finite-time stable and T is continuous on Γ and satisfies*

$$T(x) \leq \tfrac{1}{c(1-\alpha)} V(x)^{1-\alpha}, \quad \forall x \in \Gamma. \tag{1.15}$$

If, in addition $\Gamma = \mathbb{R}^n$, V is proper and $\dot{V}$ takes negative values on $\Gamma\backslash\{0\}$, then the origin is a globally finite-time stable equilibrium of (1.1).

In fact, under some mild additional assumptions, (1.14) is also a necessary condition for FTS [Moulay and Perruquetti, 2006][1], the proof of this result is based on a converse Lyaponv theorem, which is the subject of the next section.

Theorem 1.3 (Polyakov, 2012). *Suppose that there exists a continuous radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and real numbers $c_1, c_2 > 0$, $\alpha \in (0,1)$ and $\beta > 1$ such that*

$$\dot{V}(x) \leq -c_1 V(x)^{\alpha} - c_2 V(x)^{\beta}.$$

Then the origin is a fixed-time stable equilibrium of (1.1) *and T is continuous on Γ and satisfies*

$$T(x) \leq \tfrac{1}{c_1(1-\alpha)} + \tfrac{1}{c_2(\beta-1)}, \quad \forall x \in \mathbb{R}^n. \tag{1.16}$$

Converse Lyapunov Theorems

Converse theorems are fundamental aspects of mathematics. They complete the characterization of mathematical properties and they often reveal more knowledge about the properties being studied. In dynamical systems theory, the Lyapunov's direct method (LDM) gives sufficient conditions for

[1] see Theorem 1.6 of this thesis.

Lyapunov stability. This is, if there exists a function V as described in Theorem 1.1, then the system's origin is Lyapunov stable. It is not stated if the converse is true, this is, if all systems with a LS origin posses a Lyapunov function. The results that answered this question were given more than 50 years after the publication of LDM and it was through the development of converse results that I.G. Malkin discovered that, in the nonautonomous case, the conditions of LDM are sufficient not only to ascertain asymptotic stability but also uniform asymptotic stability [Malkin, 1954].

The first general converse theorem on LS for nonautonomous systems was given by K. Persidskii in [Persidskii, 1937], where he obtained a locally Lipschitz Lyapunov function. The first converse result on AS for both autonomous and periodic systems was given in 1949 by J.L. Massera, in his result, he obtained a continuously differentiable LF [Massera, 1949]. Around the same time, E.A. Barbashin demonstrated that for autonomous systems with an AS origin, there exists a LF with the same regularity as that of the vector field $f \in \mathcal{C}^k$, with $k \geq 1$ [Barbashin, 1950]. N.N. Krasovskii and E.A. Barbashin showed in [Barbashin and Krasovskii, 1952] that the radially unboundedness of V is a necessary and sufficient condition for AS to hold for any[2] $x \in \Gamma$, this result allowed J.L. Massera and J. Kurzweil to obtain, independently, much stronger theorems involving smooth Lyapunov functions for general nonautonomous systems [Kurzweil, 1956; Massera, 1956]. While Massera considered locally Lipschitz vector fields, Kurzweil relaxed the regularity to only continuity, thus allowing nonuniqueness of solutions.

For the scope of this thesis, and because of the former feature, we will make use of Kurzweil's theorem, adapted to the autonomous case[3].

Theorem 1.4 (Kurzweil, 1956). *If the origin of* (1.1) *is asymptotically stable on* Γ, *then there exists a smooth strict Lyapunov function* $V : \Gamma \to \mathbb{R}_{\geq 0}$ *for* (1.1).

The proof of Kurzweil's theorem works by first constructing a continuously differentiable Lyapunov function V^* and then applying smoothing techniques until finally a $\mathcal{C}^\infty$ Lyapunov function V is obtained, therefore, without loss of generality we can assume that if the origin of (1.1) is AS, then it admits an at least $\mathcal{C}^1$ strict LF.

Whenever a system has an asymptotically stable equilibrium at the origin and a strict Lyapunov function is known, the mapping $[0, +\infty) \to V(x)$, $t \overset{\varphi}{\mapsto} V(\psi_x(t))$ is well defined, differentiable and strictly decreasing. This last property means that an inverse mapping $s \overset{\theta}{\mapsto} t$, also strictly decreasing and differentiable, can be uniquely defined and satisfies

$$\theta'(s) = \frac{1}{\dot{V}(\psi_x(\theta(s)))}.$$

[2]This property is commonly referred as asymptotic stability *in the large*. Note that all the statements of Definition 1.1 are given in this sense, this is in contrast to other stability definitions where the domain for which the stability properties hold is assumed to exist. In theorems where the stated properties hold *in the large i.e* for all $x \in \Gamma$, the existence of the domain Γ, fixed *a priori*, is guaranteed through the fulfillment of the theorem's conditions.

[3]An extensive historical account of the development of converse results can be found in [Kellett, 2015].

Thus, the following equality holds:

$$T(x) = \int_0^{T(x)} dt = \int_{V(x)}^0 \theta'(s)ds = \int_0^{V(x)} \frac{ds}{-\dot{V}(\psi_x(\theta(s)))}. \tag{1.17}$$

Equation (1.17) reveals a link between the Lyapunov function V and the settling-time T. This equality allows to obtain necessary and sufficient conditions for finite-time stability and it is at the core of several other results that will be developed in this thesis.

Theorem 1.5 (Moulay and Perruquetti, 2006). *Let us consider system* (1.1) *and suppose additionally that f is locally Lipschitz outside the origin. The following properties are equivalent:*

i) the origin of system (1.1) *is finite-time stable on* Γ.

ii) there exists a class-$\mathcal{C}^\infty$ strict Lyapunov function $V : \Gamma \to \mathbb{R}_{\geq 0}$ for system (1.1), *satisfying for all $x \in \Gamma$*

$$\int_{V(x)}^0 \frac{ds}{\dot{V}(\psi_x(\theta(s)))} < +\infty,$$

where the map $s \overset{\theta}{\mapsto} t$ fulfills the identity $s = V(\psi_x(\theta(s)))$.

Moreover if i) or ii) are verified, all smooth Lyapunov functions $V : \Gamma \to \mathbb{R}_{\geq 0}$ for the system (1.1) *satisfy for all $x \in \Gamma$*

$$T(x) = \int_{V(x)}^0 \frac{ds}{\dot{V}(\psi_x(\theta(s)))} < +\infty.$$

Notice however, that in this last theorem the regularity of the settling-time function has not been taken into account. In this regard, the following converse theorem involves finite-time stability with continuous settling-time.

Theorem 1.6 (Bhat and Bernstein, 2000; Moulay and Perruquetti, 2006). *Suppose that the origin of* (1.1) *is finite-time stable on* Γ *and that the settling-time function T is continuous at 0. Then there exist $\alpha \in (0,1)$, $c > 0$ and a continuous function $V : \Gamma \to \mathbb{R}_{\geq 0}$ such that $\dot{V}$ is real valued, continuous on* Γ *and satisfies*

$$\dot{V}(x) \leq -cV(x)^\alpha, \quad \forall x \in \Gamma.$$

Let us remark that by assuming only continuity of T at zero in Theorem 1.6, the conclusion about the regularity of V cannot be strengthen to, for example, Hölder continuity. To see this, consider the equation (1.15). It establishes a regularity relation between V and T and shows that if V is Hölder continuous at the origin, then T has to be Hölder continuous at the origin. On the other hand it would not be reasonable to assume Hölder continuity of the settling-time function since it is known that there exist FTS systems with continuous (not Hölder) settling-time functions (see [Bhat and Bernstein, 2000, Example 2.3]).

1.2 Input-to-State Stability

This section addresses the robustness of nonlinear systems. The type of robustness to be discussed relates the magnitude of the input of a given system with a magnitude on the system's states. It can be roughly enunciated as "regardless of the system's initial state, if the inputs are uniformly small, then the state must eventually be small" [Sontag, 1998]. This property, known as *input-to-state stability* (ISS), is now a standard tool for robustness analysis.

Since we now deal with input signals, we will adopt the following notation: for a Lebesgue measurable function $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, we use $\|d\|_{[t_0,t_1)}$ as $\operatorname{ess\,sup}_{t\in[t_0,t_1)}\|d(t)\|$ to define the norm of $d(t)$ in the interval $[t_0, t_1)$. Then the set of essentially bounded and measurable functions $d(t)$ with the property $\|d\|_{[0,+\infty)} < +\infty$ is denoted as $\mathcal{L}_\infty$ and $\mathcal{L}_D = \{d \in \mathcal{L}_\infty : \|d\|_{[0,+\infty)} \leq D\}$ for any $D > 0$.

We consider autonomous systems with inputs of the form

$$\dot{x} = f(x, d), \quad t \geq 0, \tag{1.18}$$

where $x \in \mathbb{R}^n$ is the state and $d(t) \in \mathbb{R}^m$ is the input, $d \in \mathcal{L}_\infty$; $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuous and ensures forward existence of the system solutions, at least locally in time, and $f(0,0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and an input $d \in \mathcal{L}_\infty$, the corresponding solution is denoted by $\psi_{x_0}(t,d)$ for any $t \geq 0$ for which the solution exists. Since f might not be Lipschitz in x and/or in d, (1.18) might not have unique solutions. In this work we are interested in strong stability notions *i.e.* properties satisfied for all solutions. Thus, with a slight inexactness in the notation, we will assume that if a property is satisfied for all initial conditions in a set, then it will also hold for all solutions issued from those initial conditions.

Example 1.3

To motivate the importance of robustness analysis in nonlinear systems, consider the scalar system

$$\dot{x} = -3x + (1 + 2x^2)d.$$

It has a globally exponentially stable origin when $d = 0$. However, if $x(0) = 2$ and $d(t) = 1$, the solution

$$\psi_x(t) = \frac{3 - e^t}{3 - 2e^t}$$

is unbounded and has a finite escape time.

The ISS definition relies on comparison functions, which are briefly recalled in what follows. A continuous function $\vartheta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a *class-$\mathcal{K}$ function* if it is strictly increasing with $\vartheta(0) = 0$; ϑ is a *class-$\mathcal{K}_\infty$ function* if it is a class-$\mathcal{K}$ function and $\vartheta(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the *class-$\mathcal{KL}$* if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_{\geq 0}$ and $\lim_{t\to+\infty} \beta(s,t) = 0$ for each fixed $s \in \mathbb{R}_{\geq 0}$. We are now ready to present the definition of ISS.

Definition 1.3 (Sontag and Wang, 1995). The system (1.18) is called *input-to-state stable (ISS)*, if for any input $d \in \mathcal{L}_\infty$ and any $x_0 \in \mathbb{R}^n$ there exist some functions $\beta \in \mathcal{KL}$, $\vartheta \in \mathcal{K}$ such that

$$\|\psi_{x_0}(t,d)\| \leq \beta(\|x_0\|,t) + \vartheta(\|d\|_{[0,t)}) \quad \forall t \geq 0. \tag{1.19}$$

The function ϑ is called the nonlinear gain.

As can be seen, ISS relates a norm in the state with a gain (as a comparison function) in the input. The less the nonlinear gain is, the less impact the input has in the system's state and the more robust the system is.

Lyapunov Characterization of ISS systems.

As with stability analysis, ISS can be characterized using Lyapunov functions. In the context of systems with inputs, to stress the dependence on the variables x and d, instead of using the notation $\dot{V}(x)$ to denote the derivative of $V(x)$ along the trajectories of (1.18), we will sometimes use the notation $DV(x)f(x,d)$ where D is a derivative operator. As in the previous section, whenever V fails to be continuously differentiable, $DV(x)f(x,d)$ will signify the upper-right Dini derivative of V along the trajectories of (1.18).

Definition 1.4 (Sontag and Wang, 1995). A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called *ISS Lyapunov function* for system (1.18) if for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\chi, \gamma \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{1.20}$$

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -\gamma(\|x\|). \tag{1.21}$$

The following theorem is the main result in ISS theory, it relates, in both directions, the existence of an ISS Lyapunov function with the ISS property of a given system.

Theorem 1.7 (Sontag and Wang, 1995). *The system (1.18) is ISS if and only if it admits an ISS Lyapunov function.*

Once an ISS Lyapunov function is known, the nonlinear gain ϑ in (1.19) is given by $\vartheta = \alpha_1^{-1} \circ \alpha_2 \circ \chi$ [Khalil, 2002, Theorem 4.19]. Also, without loss of generality, it is possible to assume in (1.21) that $\gamma \in \mathcal{K}_\infty$ [Lin et al., 1996, Remark 4.1].

In the case of LF for autonomous systems, a useful consequence of the continuity and positive definiteness of V is that it can be bounded by comparison functions.

Lemma 1.2 (Khalil, 2002). *Let $V : \Gamma \to \mathbb{R}$ be a continuous positive definite function. Let $\mathcal{B}(s) \subset \Gamma$ for some $s > 0$. Then, there exist class-$\mathcal{K}$ functions α_1 and α_2, defined on $[0,s]$, such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathcal{B}(s).$$

If $\Gamma = \mathbb{R}^n$, the functions α_1 and α_2 will be defined on $[0,\infty]$ and the foregoing inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class-$\mathcal{K}_\infty$.

Example 1.4 [Khalil, 2002]

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + d. \end{aligned} \tag{1.22}$$

Let us start by verifying the stability of the origin when $d = 0$ with the candidate LF

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4.$$

We have that

$$DV(x)f(x,d) = -x_1^2 + x_1x_2^2 - x_2^4 = -(x_1 - \tfrac{1}{2}x_2^2)^2 - \tfrac{3}{4}x_2^4,$$

which shows that the origin is GAS. Considering the case $d \neq 0$ and keeping the candidate function we obtain

$$DV(x)f(x,d) = -\tfrac{1}{2}(x_1 - x_2^2)^2 - \tfrac{1}{2}(x_1^2 + x_2^4) + x_2^3 d \leq -\tfrac{1}{2}(x_1^2 + x_2^4) + |x_2|^3|d|.$$

In order to dominate the term $|x_2|^3|d|$ using the term $-\frac{1}{2}(x_1^2+x_2^4)$, we rewrite the last inequality as

$$DV(x)f(x,d) \leq -\tfrac{1}{2}(1-\theta)(x_1^2 + x_2^4) - \tfrac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|d|,$$

where $\theta \in (0,1)$. The term $-\frac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|d|$ will be nonpositive if $|x_2| \geq 2|d|/\theta$ or if $|x_2| \leq 2|d|/\theta$ and $|x_1| \geq (2|d|/\theta)^2$. This condition is captured by

$$\max\{|x_1|, |x_2|\} \geq \max\left\{\tfrac{2}{\theta}|d|, \left(\tfrac{2}{\theta}|d|\right)^2\right\}.$$

Using the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ and defining the class-$\mathcal{K}$ function ρ as

$$\rho(r) = \max\left\{\tfrac{2}{\theta}r, \left(\tfrac{2}{\theta}r\right)^2\right\}$$

we see that $\|x\|_\infty \geq \rho(|d|) \Rightarrow DV(x)f(x,d) \leq -\frac{1}{2}(1-\theta)(x_1^2 + x_2^4)$ so that the condition (1.21) is satisfied. The condition (1.20) is fulfilled given the positive definiteness and the radial unboundedness of V (see Lemma 1.2), hence the system (1.22) is ISS. Figure 1.3 shows a trajectory of this system with $d = 0$ (left) and with $d(t) = 0.5\sin(5t)$. Although we remark a deviation of the trajectory in the perturbed case, the trajectory converges to a small neighborhood of the origin and remains there for any future time. Given the ISS property of the origin, this behavior will hold for any initial condition on $\mathbb{R}^2$.

As the next lemma states, there is an alternative definition of an ISS Lyapunov function that provides a *dissipativity like* characterization. In many cases, this equivalent characterization makes the calculations more tractable.

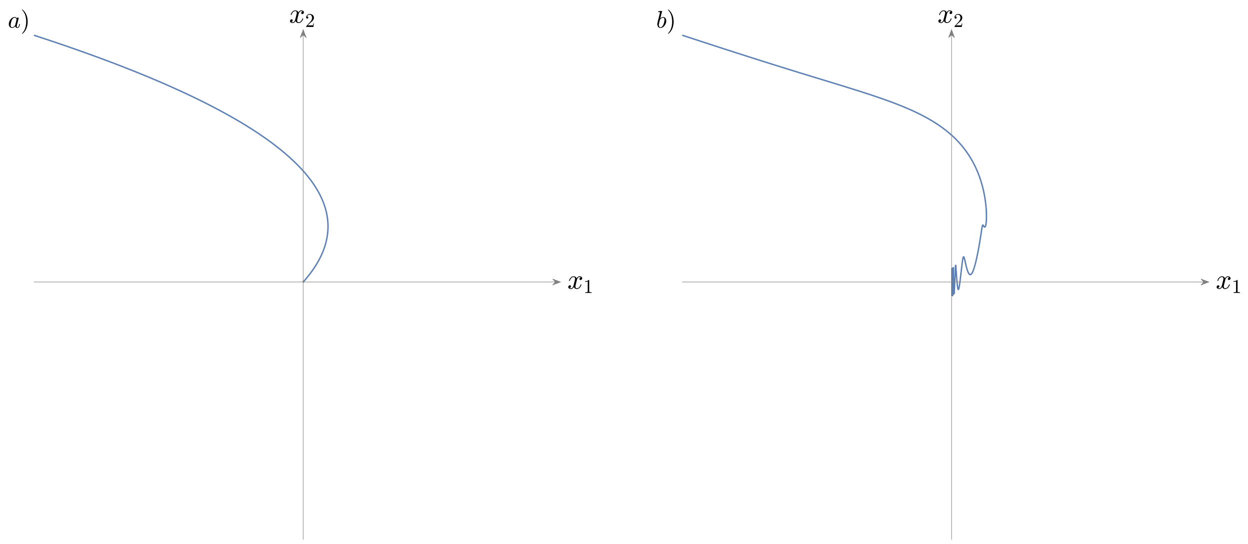

Figure 1.3 – Trajectory of system (1.22) starting from $x(0) = (-2, 2)$ a) with $d = 0$ and b) with $\mathrm{d}(t) = \frac{1}{2}\sin(5t)$.

Lemma 1.3 (Sontag and Wang, 1995). *A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS Lyapunov function for* (1.18) *if and only if there exist $\alpha_1, \alpha_2, \delta, \gamma \in \mathcal{K}_\infty$ such that* (1.20) *holds and*

$$DV(x)f(x,d) \leq \delta(\|d\|) - \gamma(\|x\|). \tag{1.23}$$

Integral Input-to-State Stability

Another type of robustness, closely related with ISS, arises when the system state remains bounded for integrally bounded inputs.

Example 1.5 (Angeli et al., 2000)

Consider the following scalar system

$$\dot{x} = -\arctan x + d \tag{1.24}$$

and let $V(x) = x \arctan x$. Then

$$\begin{aligned} DV(x)f(x,d) &= \arctan x(-\arctan x + d) + \tfrac{x}{1+x^2}(-\arctan x + d) \\ &\leq -(\arctan|x|)^2 + 2|d|, \end{aligned}$$

which does not fulfill the Lyapunov characterization of Lemma 1.3 since $(\arctan|x|)^2$ is not of class-$\mathcal{K}_\infty$. In fact system (1.24) does not have any ISS Lyapunov function because it is not an ISS system. To prove this it suffices to notice that the trajectory starting at $x(0) = 1$ with $d(t) = \pi/2$ is unbounded. However, under the metrics of integral ISS, it can be show that for integrally bounded inputs the system is nonetheless robust.

Definition 1.5 (Sontag and Wang, 1995). System (1.18) is *integral input-to-state stable* (iISS) if there exist functions $\vartheta_1, \vartheta_2 \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty(\mathbb{R}^p)$ the following estimate holds:

$$\vartheta_1(\|\psi_{x_0}(t,d)\|) \leq \beta(\|x_0\|, t) + \int_0^t \vartheta_2(\|d(s)\|)ds, \forall t \geq 0. \tag{1.25}$$

Definition 1.6 (Sontag and Wang, 1995). A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an iISS Lyapunov function for (1.18) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\delta \in \mathcal{K}$ and a positive definite continuous function γ such that (1.20) and (1.23) hold.

Let us stress that the difference between an ISS LF and a iISS one is that in ISS, the function γ belongs to the class-$\mathcal{K}$ while in the iISS case it is continuous and positive definite. As in ISS, system (1.18) is iISS if and only if there exist an iISS LF for it [Dashkovskiy et al., 2011, Theorem 3.1].

Coming back to the Example 1.5, the system (1.24) is iISS, therefore an estimate of the magnitude of the state $\psi_x(t,d)$ on the form (1.25) holds. Note that the term involving ϑ_2 can be seen as an L_2-norm, thus measuring in some sense the energy of the input.

Similar to practical stability, whenever the solutions of system (1.18) satisfy the estimate (1.18) with an adittional constant offset *i.e.*

$$\|\psi_{x_0}(t,d)\| \leq \beta(\|x_0\|, t) + \vartheta(\|d\|_{[0,t)}) + c \quad \forall t \geq 0, \tag{1.26}$$

where $c \geq 0$, we say that the system (1.1) is *input-to-state practically stable* (ISpS). Accordingly, an ISpS Lyapunov function satisfies (1.20) and

$$DV(x)f(x,d) \leq \sigma(\|d\|) - \gamma(\|x\|), \quad \forall x \in \mathbb{R}^n, \forall d \in \mathbb{R}^m, \tag{1.27}$$

where σ is a continuous and nondecreasing function and $\gamma \in \mathcal{K}_\infty$. In accordance to the main theorems on ISS, a given system is ISpS if and only if there exists a ISpS Lyapunov function for it [Sontag and Wang, 1996, Proposition 6.4][4].

In the context of observation, ISS, ISpS and iISS provide a qualitative measure of the sensitivity of the observation error with respect to disturbances, noises and perturbations.

1.3 Homogeneous Systems

Homogeneity is a symmetry-like property of mathematical objects that preserve a certain multiplicative scaling. For instance, we say that a function f that satisfies $f(\lambda x) = \lambda^k f(x)$ for any $\lambda > 0$, is homogeneous of degree $k \in \mathbb{R}$. This means that a scaling of a factor λ in the arguments of f, produces a scaling of the factor λ^k of the original function f. This strong multiplicative regularity makes homogeneous systems, in spite of being in general nonlinear, to exhibit properties that are typical of linear systems. This properties include ISS robustness [Bernuau et al., 2013], tolerance to time delays [Efimov et al., 2014], scalability of trajectories and equivalence between attractivity

[4] In [Sontag and Wang, 1996], the authors make reference to the property *comptact ISS* and they later show that this property equivalent to ISpS (see Proposition 6.3 of the same reference).

and stability [Hahn, 1967], among others. In this sense, homogeneous systems can be considered as halfway between linear and nonlinear systems.

Homogeneity as a property of symmetric polynomials was studied long ago by Euler and a more general concept of homogeneity, called weighted homogeneity, was first studied in the context of dynamical systems by V.I. Zubov [Zubov, 1958], M. Kawski [Kawski, 1990] and H. Hermes [Hermes, 1991] independently.

Standard Homogeneity

Definition 1.7 (Hahn, 1967). Let n and m be two positive integers. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be homogeneous of degree $k \in \mathbb{R}$, in the classical sense, if

$$\forall \lambda > 0 : f(\lambda x) = \lambda^k f(x).$$

The function $f(x) = Ax$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ *i.e* a linear system, is homogeneous of degree 1 since $f(\lambda x) = \lambda f(x)$.

The following examples show that homogeneous functions can be nonlinear or discontinuous

Example 1.6

⋄ The function

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is homogeneous of degree 1, nonlinear and continuous.

⋄ The function

$$f(x) = \begin{cases} \frac{\lceil x \rfloor^{1/2} + \lceil y \rfloor^{1/2}}{x+y}, & \text{if } x+y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

is homogeneous of degree $-\frac{1}{2}$ and discontinuous.

Weighted Homogeneity

Weighted homogeneity broadens the concept of standard homogeneity by allowing the multiplicative factor λ to have different powers for each coordinate. Let us start with the fundamental definition of this property. Let $r = (r_1, ..., r_n)$ be an n-tuple of positive real numbers called *weights*. $r_{\max} = \max_{1\leq j\leq n} r_j$ and $r_{\min} = \min_{1\leq j\leq n} r_j$ denote the maximum and the minimum element of r, respectively. The matrix $\Lambda_r(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$ is called the *dilation matrix* associated to the vector of weights r and is defined for all $\lambda > 0$. Note that for any $x \in \mathbb{R}^n$, $\Lambda_r(\lambda)x = (\lambda^{r_1}x_1, ..., \lambda^{r_i}x_i, ..., \lambda^{r_n}x_n)^T$. The r-homogeneous norm is denoted for any $x \in \mathbb{R}^n$ as $\|x\|_{r,\rho} = \left(\sum_{i=0}^n |x_i|^{\rho/r_i}\right)^{1/\rho}$ where $\rho \geq r_{\max}$. When the value of ρ is omitted, *i.e* $\|x\|_r$, it will be taken as $\rho = \Pi_{i=1}^n r_i$. The *unit sphere* and a *ball* in the homogeneous norm are defined as $\mathcal{S}_r = \{x \in \mathbb{R}^n : \|x\|_r = 1\}$ and $\mathcal{B}_r(s) = \{x \in \mathbb{R}^n : \|x\|_r \leq s\}$ for $s \geq 0$. Note that the homogeneous norm is not a norm in the usual sense since it does not satisfy the subadditivity property.

The next lemma shows a relevant relation between the Euclidean norm and the homogeneous norm.

Lemma 1.4 (Bernuau, 2013). *Let $r = [r_1, .., r_n]$ be a vector of weights. For any $x \in \mathbb{R}^n$, there exists some $\overline{\sigma}, \underline{\sigma} \in \mathcal{K}_\infty$ such that $\underline{\sigma}(\|x\|_r) \leq \|x\| \leq \overline{\sigma}(\|x\|_r)$ holds.*

Definition 1.8 (Bacciotti and Rosier, 2005). A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be r-homogeneous of degree $\eta \in \mathbb{R}$ if

$$V(\Lambda_r(\lambda)x) = \lambda^\eta V(x), \quad \forall x \in \mathbb{R}^n, \forall \lambda > 0.$$

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be r-homogeneous of degree $\nu \in \mathbb{R}$ if

$$f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda) f(x), \quad \forall x \in \mathbb{R}^n, \forall \lambda > 0, \tag{1.28}$$

which is equivalent to f_i being r-homogeneous of degree $\nu + r_i$, for each i.

Let us remark that there is a key difference between a homogeneous function and a homogeneous vector field. A dilated argument $\Lambda_r(\lambda)x$ produces a multiplication factor of λ^η in a homogeneous function *i.e.* $V(\Lambda_r(\lambda)x) = \lambda^\eta V(x)$, whereas the same dilated factor produces a multiplication factor of $\lambda^{\nu + r_i}$ for each vector field coordinate x_i *i.e.* $f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda) f(x)$. The system (1.1) is called an r-homogeneous system of degree ν if the vector field f is r-homogeneous of degree ν.

Example 1.7

The function $V(x) = x_1^2 + x_2^3$ is r-homogeneous of degree $\eta = 4$ with $r = (2, \frac{4}{3})$ since $V(\lambda^2 x_1, \lambda^{4/3} x_2) = \lambda^4 V(x) \ \ \forall \lambda > 0$. Figure 1.4 shows a plot of the function V, note that a scaling of the factor λ^{r_1} in the x_1 axes and a scaling of the factor λ^{r_2} in the x_2-axis produces a scaling of the factor λ^η in the V-axis. Note that $V(x)$ is also r-homogeneous of degree 2 with $r = (1, \frac{2}{3})$, this shows that the homogeneity degree and weights are not unique. In fact, it is always possible to select $r_1 = 1$ and scale η and r_i, $i \neq 1$. Notice also that the simmetry-like property of V can be seen along the curve λ^η, depicted in orange.

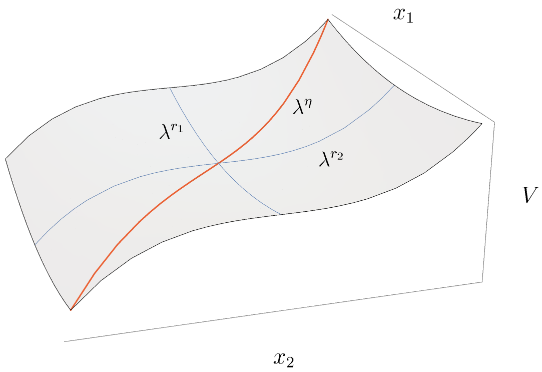

Figure 1.4 – Plot of the r-homogeneous function $V(x) = x_1^2 + x_2^3$.

Example 1.8 ***Standard Homogeneity vs Weighted Homogeneity***

The function $V(x) = x_1^2 + x_2^2$ is homogeneous of degree $k = 2$ in the standard sense. The function $W(x) = x_1^4 + x_2^2$ is not homogeneous in the standard sense but it is r-homogeneous of degree $\eta = 4$ with $r = (1,2)$. Figure 1.5 shows a plot of both functions where it is posible to see that $V(\lambda x_1, \lambda x_2) = \lambda^2 V(x_1, x_2)$ whereas $W(\lambda x_1, \lambda^2 x_2) = \lambda^4 W(x_1, x_2)$.

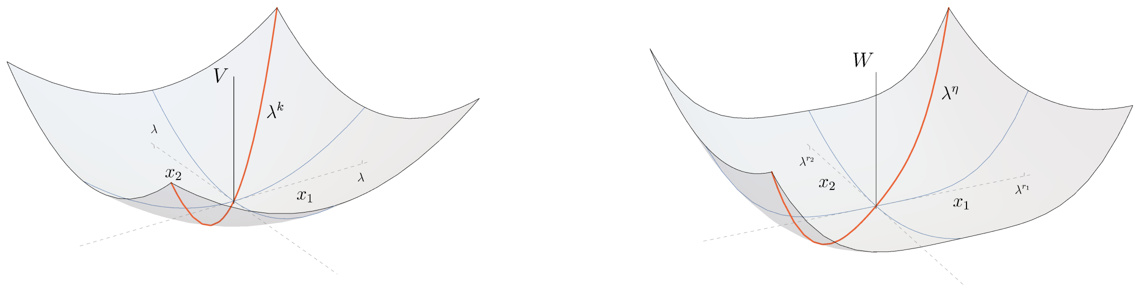

Figure 1.5 – Plot of the homogeneous function $V(x) = x_1^2 + x_2^2$ (left) and of the r-homogenous function $W(x) = x_1^4 + x_2^2$ (right).

The scaling properties of homogeneous vector fields are slightly more involved and before presenting an illustrative example, let us introduce a fundamental theorem from which many of the properties of homogeneous systems are derived.

Theorem 1.8 (Zubov, 1958). *Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is an r-homogeneous vector field of degree ν.*

i) If the curve $x(t)$ is a solution of (1.1)*, then any curve of the family $\Lambda_r(\lambda)x(\lambda^\nu t)$, where $\lambda > 0$, is also a solution of* (1.1).

ii) Any solution $\psi(x,t)$ of (1.1) *satisfies*

$$\psi(x, \lambda^\nu t) = \Lambda_r^{-1}(\lambda)\psi(\Lambda_r(\lambda)x, t) \quad \forall \lambda > 0, \forall t \geq 0. \tag{1.29}$$

Theorem 1.8 presents two distinct properties of homogeneous systems. On one hand it states that once a solution of the system is known, a whole family of solutions, parametrized by the factor

λ, can be obtained. On the other hand, each individual solution satisfies the scaling property (1.29), which can be interpreted as "a scaling in time of a solution, can be compensated by scaling the solution itself and the initial condition from which the solution originated" [Hahn, 1967].

Example 1.9

Consider the following vector field

$$\begin{aligned} \dot{x}_1 &= f_1(x) = x_2 - x_1^3, \\ \dot{x}_2 &= f_2(x) = -x_1^5. \end{aligned} \tag{1.30}$$

It is an r-homogeneous vector field of degree $\nu = 2$ with $r = (1, 3)$ since $f_1(\lambda^1 x_1, \lambda^3 x_2) = \lambda^{2+1} f_1(x_1, x_2)$ and $f_2(\lambda^1 x_1, \lambda^3 x_2) = \lambda^{2+3} f_2(x_1, x_2)$. Figure 1.6.a depicts the evolution of a single solution $\psi(x^*, t)$ of the system (1.30) starting from the initial condition $x^* = (\frac{3}{2}, 1)$. At the instant $t^* = 0.02$, the solution ψ_{x^*} reaches the point $A = \psi(x^*, t^*)$. A scaling of the factor λ^ν with $\lambda = 8$ of the time t^* will produce that the solution reaches the point $B = \psi(x^*, \lambda^\nu t^*)$. Figure 1.6.b shows that B can also be reached by scaling, instead of the time, the initial condition x^* and the solution ψ, this is, $B = \Lambda_r^{-1}(\lambda)\psi(\Lambda_r(\lambda)x^*, t^*)$. This property is satisfied for any $t \geq 0$, thus illustrating the item ii) of Theorem 1.8.
Figure 1.7.a shows a family of curves ψ_κ obtained by scaling the solution ψ as $\psi_\kappa(x, t) = \psi(\Lambda_r(\kappa)x^*, \kappa^\nu t)$ for six different values of κ. The Figure 1.7.b shows the phase portrait of system (1.30) and illustrates the fact that the family of curves ψ_κ are indeed solutions of the system (1.30).

If a vector field fails to exhibit a global degree of homogeneity but behaves as a homogeneous vector field near infinity and/or near the origin, we say that it is *locally homogeneous*.

Definition 1.9 (Andrieu et al., 2008, Bernuau et al., 2013)**.** A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is (r_0, λ_0, f_0)-*homogeneous* with degree $\eta_0 \geq -r_{0\,\min}$ (f_0 is an r_0-homogeneous vector field and $r_{0\,\min} = \min_{1 \leq j \leq n} r_{0n}$) and $\lambda_0 \in \mathbb{R}_+ \cup \{+\infty\}$ if $\lim_{\lambda \to \lambda_0} [\lambda^{-\eta_0} D_{r_0}^{-1}(\lambda) f(D_{r_0}(\lambda)x) - f_0(x)] = 0$, for all $x \in S_{r_0}$, uniformly on S_{r_0} with $\lambda_0 \in \{0, \infty\}$.

Let us now focus on the consequences that the scaling behavior of homogeneous systems have in the study of stability and robustness of such systems.

Example 1.10

Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3, \\ \dot{x}_2 &= -x_1^5 + x_2^2. \end{aligned} \tag{1.31}$$

It has an equilibrium at the origin and it is not homogeneous since it is not possible to find real numbers r_1, r_2 and ν that satisfy (1.28). When trying to find out if the origin is

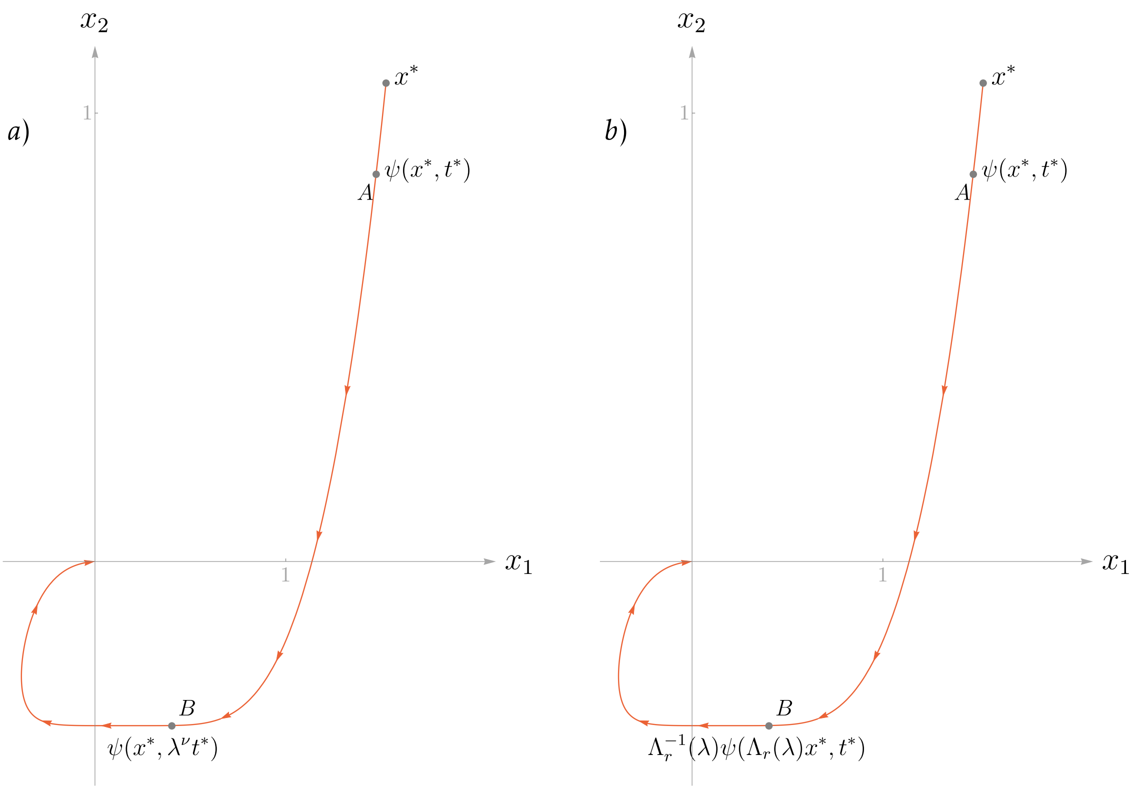

Figure 1.6 – Plot of a solution $\psi(x^*, t)$ of the vector field (1.30). a) Shows the transit from A to B by scaling t^*. b) Shows the transit from A to B by scaling x^* and ψ.

stable, not only its linearization is unstable, but its Taylor series expansion up to degree 4 is also unstable. Taking the homogeneous weights $r_1 = 1, r_2 = 3$ and eliminating the term x_2^2 produces the following homogeneous approximation of system (1.31):

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3, \\ \dot{x}_2 &= -x_1^5. \end{aligned} \tag{1.32}$$

Using the Lyapunov function $V(x) = \frac{1}{18}(4x_1^6 - 6x_1^3 x_2 + 21x_2^2)$, it can be easily shown that the origin of both the original system and its homogeneous approximation is asymptotically stable. This example aims to show that for certain systems, its homogeneous approximation captures better the system's behavior than its linearization.

An outstanding property of homogeneous systems is that many of the properties that hold locally, will immediately hold globally. As the next theorem shows, this includes stability.

Theorem 1.9 (Hahn, 1967). *Consider the homogeneous system* (1.1) *with a continuous vector field f and with forward uniqueness of solutions. If the origin is a locally attractive equilibrium, then the origin is globally asymptotically stable.*

This last result may also roughly stated as "for homogeneous systems, attractivity implies

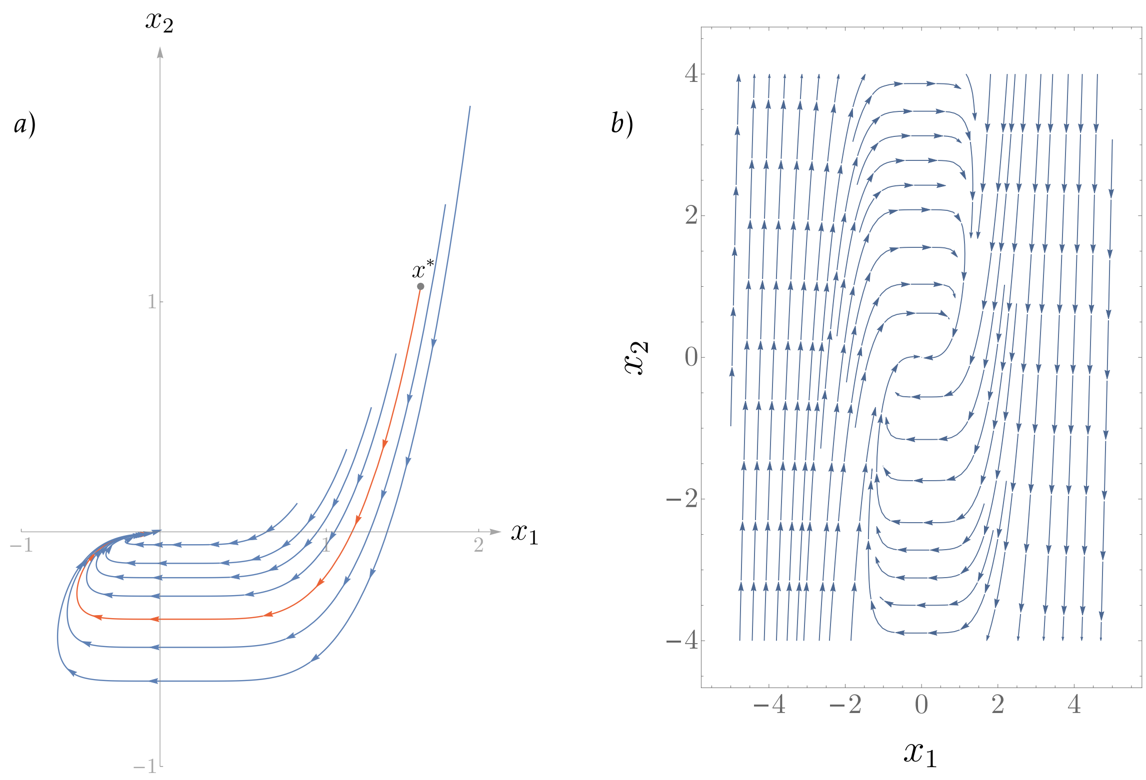

Figure 1.7 – a) Plot of the family of solutions $\psi_\kappa(x^*, t)$ of the vector field (1.30). b) Phase portrait of the system (1.30).

stability".

Regarding nonasymptotic stability rates, stable homogeneous systems also provide a characterization based only on its homogeneity degree, this is, without relying on Lyapunov analysis.

Lemma 1.5 (Nakamura et al., 2002)**.** *If the homogeneous system* (1.1) *is r-homogeneous of degree* ν *and asymptotically stable at the origin, then it is*

i) globally finite-time stable at the origin if $\nu < 0$*;*

ii) globally exponentially stable at the origin if $\nu = 0$*;*

iii) globally fixed-time[5] *stable with respect to the unit ball* $\mathcal{B}_r(1)$ *if* $\nu > 0$.

Example 1.11 (Bhat and Bernstein, 2005).

Consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lceil x_1 \rfloor^{\frac{\eta}{2-\eta}} - k_2 \lceil x_2 \rfloor^{\eta}, \end{aligned} \tag{1.33}$$

[5] In fact, in the work of H. Nakamura, the term "fixed-time" is not employed, this was a term later adopted in the literature. In the original source, item *iii*) reads "the states x converge to an arbitrary open set which includes the origin in a constant period from each initial condition.", which is equivalent to the definition of FxTS with respect to a set.

where $\eta \in (0,1)$ and k_1, k_2 are such that $s^2 + k_2 s + k_1$ is a Hurwitz polynomial. The vector field (1.33), denoted by f_η, is continuous for all $\eta > 0$, r-homogeneous of degree $\frac{\eta-1}{\eta}$ with $r = (\frac{2-\eta}{2}, \frac{1}{2})$ and for $\eta = 1$ the following linear system is obtained:

$$\dot{x} = f_1(x) = Hx, \quad H = \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix}. \tag{1.34}$$

Since H is a Hurwitz matrix, the origin of (1.34) is exponentially stable and by Kurzweil's Theorem (Theorem 1.4), there exists a strict Lyapunov function V such that $\frac{\partial V(x)}{\partial x} f_1(x)$ is continuous and negative definite. Let $\mathcal{A} = V^{-1}([0,1])$, then $\mathcal{A}$ and its boundary $\partial\mathcal{A}$ are compact since V is proper and $0 \notin \partial\mathcal{A}$ since V is positive definite. Let us define a function $\varphi : (0,1] \times \partial\mathcal{A} \to \mathbb{R}$ by $\varphi(\eta, x) = \frac{\partial V}{\partial x} f_\eta(x)$. Then φ is continuous and satisfies $\varphi(1,z) < 0$ for all $z \in \partial\mathcal{A}$, *i.e.* $\varphi(\{1\} \times \partial\mathcal{A}) \subset (-\infty, 0)$. Since $\partial\mathcal{A}$ is compact, it follows that there exists some $\epsilon > 0$ such that $\varphi((1-\epsilon, 1] \times \partial\mathcal{A}) \subset (-\infty, 0)$ (see, for example, Lemma 5.8 in [Munkers, 1975]). Then, for any $\eta \in (1-\epsilon, 1)$, φ takes negative values on $\partial\mathcal{A}$ and therefore $\mathcal{A}$ is a positively invariant set of f_η for any $\alpha \in (1-\epsilon, 1)$. It follows from Theorem 1.9 that the origin of (1.33) is GAS for any $\eta \in (1-\epsilon, \epsilon)$. Finally, using Theorem 1.5 and the fact that the degree of homogeneity of f_η is negative with respect to r, we conclude that the origin of (1.33) is, furthermore, finite-time stable.

Notice that in this example we used linear systems techniques and homogeneity properties of the system to conclude FTS. In [Bhat and Bernstein, 1998] the same conclusion about the system is obtained using an explicit Lyapunov function for (1.33) but it is not known if it can be extended for higher dimensional systems.

Later in this work, we will present an implicit Lyapunov function that also asserts FTS and that, moreover, can be easily extended to an n-dimensional chain of integrators.

As for ISS, homogeneous systems also posses a convenient property. If the origin is GAS for input zero, then the homogeneity degree will determine its robustness.

Define $\tilde{f}(x,d) = [f(x,d)^T 0_m]^T \in \mathbb{R}^{n+m}$. It is an extended auxiliary vector field for the system (1.18), where 0_m is the zero vector of dimension m.

Theorem 1.10 (Bernuau et al., 2013). *Let the vector field $\tilde{f}$ be homogeneous with the weighs $r = [r_1, \dots, r_n]^T > 0$, $\tilde{r} = [\tilde{r}_1, \dots, \tilde{r}_m]$ of degree $\nu \geq -r_{\min}$, i.e. $f(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)d) = \lambda^\eta \Lambda_r(\lambda) f(x,d)$ for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$. Assume that the system* (1.18) *is globally asymptotically stable for $d = 0$, then the system* (1.18) *is*

i) ISS if $\tilde{r}_{\min} > 0$;

ii) iISS if $\tilde{r}_{\min} = 0$ and $\eta \leq 0$.

In the case of locally homogeneous systems, ISS stability can be asserted through their homogeneous approximations at 0 and at ∞.

Theorem 1.11 (Andrieu et al., 2008). *Let the vector field $\tilde{f}$ be continuous and $((r^\infty, \bar{r}^\infty), +\infty, \tilde{f}_\infty)$-homogeneous with the weights $r^\infty = (r_1^\infty, \ldots, r_n^\infty) > 0$, $\bar{r}^\infty = (\bar{r}_1^\infty, \ldots, \bar{r}_p^\infty) > 0$ and $((r^0, \bar{r}^0), 0, \tilde{f}_0)$-homogeneous with the weights $r^0 = (r_1^0, \ldots, r_n^0) > 0$, $\bar{r}^0 = (\bar{r}_1^0, \ldots, \bar{r}_p^0) > 0$. If the origins of the systems $\dot{x} = \tilde{f}_0(x, 0), \dot{x} = \tilde{f}(x, 0)$ and $\dot{x} = \tilde{f}_\infty(x, 0)$ are globally asymptotically stable then the system* (1.18) *is ISS.*

1.4 The Implicit Lyapunov Function Approach

In calculus, analytical geometry and dynamical systems theory the role of implicit functions is highly important and many fundamental theorems rely on them. An implicit function is given as a solution of an equation of the form $G(x, y) = 0$, instead of its explicit form $y = g(x)$. To find the function g one has to *solve* the equation $G(x, y) = 0$ for y, provided that a solution exists. Take, for instance, the classic example of the unitary circle. The equation $G(x, y) = x^2 + y^2 - 1 = 0$, where $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, describes a circle of radius 1 in $\mathbb{R}^2$. One solution for $G(x, y) = 0$ is given by $y = \sqrt{1 - x^2}$, a second one is given by $x = \sqrt{1 - y^2}$ and therefore $G(x, y) = 0$ implicitly defines two lower dimensional functions $g : [-1, 1] \to (0, 1]$, $g(x) = \sqrt{1 - x^2}$ and $h : [-1, 1] \to [-1, 0)$, $h(y) = \sqrt{1 - y^2}$. In contrast, $G(x, y) = x^2 + y^2 = 0$ is only satisfied by the pair of values $x = 0$, $y = 0$ and $G(x, y) = x^2 + y^2 + c = 0$ is not satisfied by any real valued solution with $c > 0$. The implicit function theorem, whose first formal proof is attributed to Augustin Cauchy, states the conditions under which there exists a solution of the equation $G(x, y) = 0$ and therefore an explicit function $y = g(x)$ or $x = h(y)$.

Theorem 1.12 ***[Implicit Function Theorem]*** *(Khalil, 2002). Assume that a function $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable at each point (x, y) of an open set $S \subset \mathbb{R}^n \times \mathbb{R}^m$. Let (x_0, y_0) be a point in S for which $G(x_0, y_0) = 0$ and for which the Jacobian matrix $\left[\frac{\partial G}{\partial y}\right](x_0, y_0)$ is nonsingular. Then there exist neighborhoods $U \subset \mathbb{R}^n$ of x_0 and $Y \subset \mathbb{R}^m$ of y_0 such that for each $x \in U$, the equation $G(x, y) = 0$ has a unique solution $y \in Y$. Moreover, this solution can be given as $y = g(x)$, where g is continuously differentiable at $x = x_0$.*

It is often the case that to study the function y, it is better to work with its implicit representation $G(x, y) = 0$ rather than with its explicit form $y = g(x)$.

Example 1.12 (Courant and John, 2012).

The equation of the *lemniscate* curve is given by

$$F(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0$$

and it is not easily solved for y. For $x = 0, y = 0$ we obtain $F = 0, \partial_x F = 0, \partial_y F = 0$ and therefore the conditions of the implicit function theorem are not satisfied at the origin. This can be expected since the curve crosses *twice* trough the origin (see Figure 1.8). However, for the rest of the curve points, Theorem 1.12 can be applied and using the chain rule we have that

$$y' = -\frac{\partial_x F}{\partial_y F} = -\frac{4x(x^2+y^2) - 4a^2 x}{4y(x^2+y^2) + 4a^2 y}.$$

From this last expression it is possible to calculate the maximum and minimum values since they will occur whenever $y' = 0$ *i.e.* at $x = 0$ or at $x^2 + y^2 = a^2$. Thus the maximum values will be attained at $(\pm a\sqrt{3}/2, a/2)$ and its minimum ones at $(\pm a\sqrt{3}/2, -a/2)$. Note that this critical points were obtained without calculating an explicit representation of the function y.

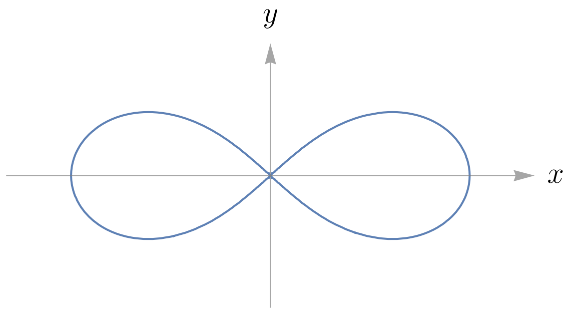

Figure 1.8 – Lemniscate curve.

It is worth to mention that $G(x,y) = 0$ does not necessarily imply an algebraic equation, as has been the case so far; it can be an integro-differential equation, a trigonometric equation, a functional equation, etc. [Courant and John, 2012]. Indeed, the shape that an implicit function can take is quite varied.

The notions about implicit functions can be extended to Lyapunov analysis. The next theorem, presented first in the work of Adamy, shows what can be asked for a Lyapunov candidate function, defined in the implicit form $Q(V,x) = 0$, in order to show global asymptotic stability of the origin.

Theorem 1.13 (Adamy, 2005; Polyakov et al., 2015). *If there exists a continuous function* $Q : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$, $(V,x) \mapsto Q(V,x)$ *satisfying the conditions*

C1 *Q is continuously differentiable on* $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \backslash \{0\}$;

C2 *for any* $x \in \mathbb{R}^n \backslash \{0\}$ *there exists* $V \in \mathbb{R}_{\geq 0} : Q(V,x) = 0$;

C3 *for* $\Omega := \left\{ (V,x) \in \mathbb{R}^{n+1} : Q(V,x) = 0 \right\}$ *we have*

$$\lim_{\substack{x \to 0 \\ (V,x) \in \Omega}} V = 0, \quad \lim_{\substack{V \to 0^+ \\ (V,x) \in \Omega}} \|x\| = 0 \text{ and } \lim_{\substack{\|x\| \to \infty \\ (V,x) \in \Omega}} V = +\infty;$$

C4 $-\infty < \partial_V Q(V,x) < 0$ *for* $V \in \mathbb{R}_{\geq 0}$ *and* $x \in \mathbb{R}^n \backslash \{0\}$;

C5 $\partial_x Q(V,x) f(x) < 0$ *for all* $(V,x) \in \Omega$,

then the origin of (1.1) *is globally uniformly asymptotically stable and the function* $Q(V,x) = 0$ *implicitly defines a Lyapunov function* $V(x)$ *for* (1.1).

If Q satisfies conditions **C1**-**C5**, then it is called an *implicit Lyapunov function* (ILF). Clearly, conditions **C1** and **C4** are required to satisfy the implicit function theorem. Conditions **C2** and **C3** ensure that $Q(V,x)=0$ defines implicitly a unique, continuously differentiable radially unbounded positive definite function V. The last condition is the implicit version of the differential inequality of the Lyapunov's direct method.

With the Lyapunov characterization of finite-time and fixed-time stability (Theorems 1.2 and 1.3) the following results are straightforward.

Theorem 1.14 (Polyakov et al., 2015). *Suppose that there exists a function Q that satisfies conditions* **C1**-**C4** *of Theorem 1.13 and*

C6 $\partial_x Q(V,x)f(x) \leq cV^{\alpha}\partial_V Q(V,x)$, $\forall (V,x)\in\Omega$, *where $c>0$ and $\alpha\in(0,1)$ are some constants.*

Then the origin of (1.1) *is uniformly finite-time stable with the settling-time estimate* $T(x_0) \leq \frac{V_0^{1-\alpha}}{c(1-\alpha)}$ *where* $V_0\in\mathbb{R}_{\geq 0} : Q(V_0,x_0)=0$.

For the fixed-time case, we present two implicit Lyapunov characterizations. The first one makes use of two functions Q_1 and Q_2 that implicitly define a single Lyapunov function V. The second one presents a more succint condition by using a single implicit function Q; nonetheless this condition may be more difficult to verify (see Chapter 4).

Theorem 1.15 (Polyakov et al., 2015). *Let two functions $Q_1(V,x)$ and $Q_2(V,x)$ satisfy conditions* **C1**-**C4** *of Theorem 1.13 and*

C7 $Q_1(1,x)=Q_2(1,x)$ *for all* $x\in\Gamma\backslash\{0\}$.

C8 $\partial_x Q_1(V,x)f(x)\leq c_1V^{1-\mu}\partial_V Q_1(V,x)$, $\forall V\in(0,1], \forall x\in\{z\in\mathbb{R}^n\backslash 0 : Q_1(V,z)=0\}$, *where $c_1>0$ and $\mu\in(0,1]$ are some constants.*

C9 $\partial_x Q_2(V,x)f(x)\leq c_2V^{1+\kappa}\partial_V Q_2(V,x)$, $\forall V\geq 1, \forall x\in\{z\in\mathbb{R}^n\backslash\{0\} : Q_2(V,z)=0\}$, *where $c_2>0$ and $\kappa>0$ are some constants.*

Then the origin of (1.1) *is fixed-time stable with continuous T, satisfying* $T(x_0)\leq\frac{1}{c_1\mu}+\frac{1}{c_2\kappa}$.

Theorem 1.16. *Suppose that there exists a function Q that satisfies all conditions of Theorem 1.13 and*

C10 $\partial_V Q(V,x)f(x)\leq(c_1V^{\alpha}+c_2V^{\beta})\partial_x Q(V,x)$, $\forall(V,x)\in\Omega$, *where $c_1,c_2>0$, $\alpha\in(0,1)$ and $\beta>1$ are some constants.*

Then the origin of (1.1) *is fixed-time stable with continuous T, satisfying* $T(x_0)\leq\frac{1}{c_1(1-\alpha)}+\frac{1}{c_2(\beta-1)}$.

Clearly, the condition **C10** of Theorem 1.16 implies the conditions **C7**-**C9** of Theorem 1.16, however, the converse does not hold in general.

1.5 General Problem Statement

To conclude this chapter, let us present in more detail some of the questions that are set to be answered in this thesis.

⋄ Consider the system (1.1). Through Theorems 1.2, 1.5 and 1.6 we know necessary and sufficient conditions for FTS of systems having or not continuous T. Regarding FxTS, only the sufficient condition of Theorem 1.3 is known. Moreover, it has not been established if the conditions for FTS and FxTS can be stated in a more general form, that is, if they can be expressed as differential inequalities that involve a general function r that depends on V. If this is the case, what should be the regularity of r and what properties should be satisfied? This is the central topic of Chapter 2, where conditions for fixed-time stabilizability of nonlinear systems will also be presented.

⋄ Consider now the system (1.18). When $d = 0$, we have seen that it is possible, using Lyapunov analysis, to determine if the origin is AS, FT or FxTS. We have also discussed how the implicit approach can be used to obtain the same conclusions. When $d \neq 0$ the ISS framework, along with its Lyapunov characterization, is readily available. The ISS framework is built upon the consideration of trajectories of asymptotic nature, that is, if the origin of a system is ISS stable, when $d = 0$ asymptotic stability of the origin follows. A natural extension of ISS would be to investigate the properties of an ISS system whose trajectories behave *nonasymptotically*. Chapter 3 addresses this case and presents both the explicit and the implicit approach. Then, it will be possible to establish the stability rate of a system and its ISS robustness with a single Lyapunov function.

⋄ Consider finally the linear control system

$$\begin{aligned} \dot{x} &= Ax + bu(t) + d(t) \\ y &= Cx + v(t), \end{aligned} \tag{1.35}$$

where $A \in \mathbb{R}^{n\times n}$, $u(t) \in \mathbb{R}^p$ is the control input, $y(t) \in \mathbb{R}^q$ is the measured output, $d(t) \in \mathbb{R}^n$ is an exogenous disturbance, $v(t)$ is the measured noise and C and b are matrices of compatible dimensions. Suppose that we wish to design a control signal $u(t)$ and a dynamic observer

$$\dot{z} = Az + bu(t) + \hat{g}(y(t) - Cz), \quad \hat{g} : \mathbb{R}^q \to \mathbb{R}^n, \tag{1.36}$$

that will respectively, drive the states to zero in fixed-time in the absence of disturbance and estimate the state x in fixed-time in the absence of disturbances and noises. Note that using linear techniques, the most we can hope for is to assert exponential stability of the states and of the observation error. To assert FxTS, the nonlinear tools introduced in this chapter can be used.

In Chapter 4, the general MIMO linear case of system (1.35) will be studied. Here we face the lack of explicit Lyapunov functions that can provide NonA rates. Here, the implicit approach will be used to assert FTS and FxTS through the fulfillment of Linear Matrix Inequalities (LMI). This methodology will also allow to obtain estimates on the settling-time function. In all cases, we will show that the control setups are robust in an ISS sense.

In Chapter 5, we consider the case where the matrix A is in upper diagonal form *i.e.* the system (1.35) becomes a chain of integrators. The control and estimation goals will be achieved through homogeneous properties. More precisely, a controller and an observer that switch the homogeneous degree of the whole system from positive to negative will enforce FxTS. The use of homogeneous techniques greatly simplifies the stability analysis; however this comes with the price of not being able to derive settling-time estimates. Therefore, using the implicit approach, a parameter optimization analysis will be performed in order to influence the settling-time estimate.

Chapter 2

Conditions for Fixed-Time Stability

Contents

This chapter derives, using Lyapunov analysis, the conditions that render the origin of a system fixed-time stable. General necessary and sufficient conditions for FxTS that do not take into account the regularity of the settling-time function will be developed first. Next, a characterization of the FxTS property using a pair of functions will be proposed. This characterization involves a strict Lyapunov function V and a continuous positive definite function that depends on V. It will be seen that this characterization allows to obtain a sufficiency condition that rules out the case of discontinuous $T(x)$. More constructive conditions for FxTS, with specific definitions of the characterizing functions will follow and in order to obtain a converse result, the concept of *uniform* FxTS will be introduced. In the last section, the results obtained will be extended to present a sufficient condition for fixed-time stabilization of nonlinear affine systems.

Let us first introduce an example that illustrates the behavior of a fixed-time stable system.

Example 2.1

Consider the scalar system

$$\dot{x} = -\lceil x \rfloor^{\frac{1}{2}} - \lceil x \rfloor^{\frac{3}{2}}, \quad x \in \mathbb{R}. \tag{2.1}$$

The trajectories of this dynamics, for any initial condition $x_0 \in \mathbb{R}$ and any $t \geq 0$, can be obtained by direct integration and are given by

$$\psi_{x_0}(t) = \begin{cases} \tan[\arctan(|x_0|^{\frac{1}{2}}) - \frac{1}{2}t]^2 \operatorname{sign}(x_0) & \text{if } 0 \leq t \leq T(x_0) \\ 0 & \text{if } t > T(x_0) \end{cases}.$$

It is clear from the system's solutions that for any $t \geq 2\arctan(|x_0|^{\frac{1}{2}})$, $\psi_{x_0}(t) = 0$ so that the settling-time function is $T(x_0) = 2\arctan(|x_0|^{\frac{1}{2}})$. Moreover, this function is gobally bounded on $\mathbb{R}$ since $\sup_{x_0\in\mathbb{R}} T(x_0) = \pi$. Therefore, all trajectories, regardless of the initial condition, converge exactly to zero in $t \leq \pi$ seconds (see Figure 2.1).

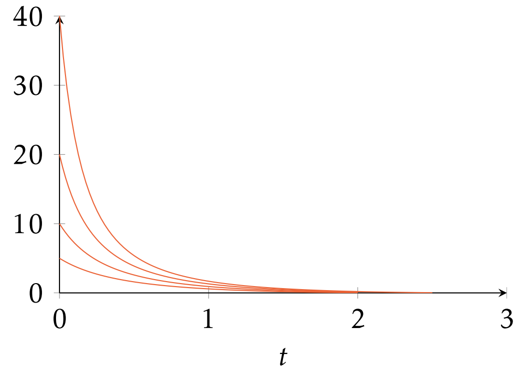

Figure 2.1 – Plot of the solutions $\psi_{x_0}(t)$ of system (2.1) for four different initial conditions.

Our aim is to deduce the conditions that a general nonlinear system has to satisfy in order to behave as (2.1), without explicitly calculating its trajectories.

Since in Example 2.1 the trajectories of the system can be calculated explicitly, so does the settling-time function. Although this is in general not possible, it will be seen that through Lyapunov analysis the existence of $T(x)$ can be proven and furthermore a finite bound for it can be obtained.

The system (2.1) possesses yet another property. Note that since $T(x_0)$ is an increasing function and $\lim_{x_0\to+\infty} T(x_0) = \lim_{x_0\to-\infty} T(x_0) = \pi$, the supremum over $x_0 \in \mathbb{R}$ of $T(x_0)$ is independent on the direction at which x_0 tends to infinity. This property will be called *uniform fixed-time stability* and it is defined as follows:

Definition 2.1. The origin of system (1.1) is called *uniformly fixed-time stable* if it is fixed-time stable on Γ and there exists some $T_m > 0$ such that

$$\liminf_{x\to\partial\Gamma} T(x) = \limsup_{x\to\partial\Gamma} T(x) = T_m.$$

If $\Gamma = \mathbb{R}^n$ and $T(x)$ satisfies

$$\liminf_{\|x\|\to\infty} T(x) = \limsup_{\|x\|\to\infty} T(x) = T_m,$$

then the origin of (1.1) is called *globally uniformly fixed-time stable*.

2.1 Conditions for Fixed-Time Stability

Our first result presents a necessary and sufficient condition for fixed-time stability, here and in the theorems that follow the properties of strict Lyapunov functions (see Definition 1.2) play a crucial role.

Theorem 2.1. *Consider system* (1.1)*. The following properties are equivalent:*

i) The origin is fixed-time stable on Γ*.*

ii) There exists a strict Lyapunov function V *for system* (1.1) *satisfying for all* $x \in \Gamma$

$$\sup_{x\in\Gamma} \int_{V(x)}^{0} \frac{ds}{\dot{V}(\psi_x(\theta(s)))} < +\infty, \tag{2.2}$$

where $s \overset{\theta}{\mapsto} t$ *is the inverse mapping of* $t \mapsto V(\psi_x(t))$*.*

Proof. $i) \Rightarrow ii)$. If the system (1.1) is fixed-time stable, then its settling-time function is such that $T_* := \sup_{x\in\Gamma} T(x) < +\infty$ and $\psi_x(t) = 0$ for all $t \geq T_*$ and for all $x \in \Gamma$. Since fixed-time stability implies asymptotic stability, according to Theorem 1.4, there exists a strict Lyapunov function V for (1.1) and therefore there exists a well defined application $[0, T(x)) \to (0, V(x)]$, $t \mapsto V(\psi_x(t))$ strictly decreasing and differentiable for all $t \in [0, T(x))$. Hence, for any $x \in \Gamma$, there exists a differentiable inverse mapping $(0, V(x)] \to [0, T(x))$, $s \overset{\theta}{\mapsto} t$, also decreasing that satisfies for all $s \in (0, V(x)]$

$$\theta'(s) = \frac{1}{\dot{V}(\psi_x(\theta(s)))}.$$

The change of variables $s = V(\psi_x(t))$ and the fact that $V(\psi_x(T(x))) = 0$ for all $x \in \Gamma$ lead to

$$T(x) = \int_0^{T(x)} dt = \int_{V(x)}^{0} \theta'(s)ds = \int_{V(x)}^{0} \frac{ds}{\dot{V}(\psi_x(\theta(s)))}.$$

Then we have that

$$+\infty > \sup_{x\in\Gamma} T(x) \geq \sup_{x\in\Gamma} \int_{V(x)}^{0} \frac{ds}{\dot{V}(\psi_x(\theta(s)))}. \tag{2.3}$$

for all $x \in \Gamma$ and the conclusion readily follows.
$ii) \Rightarrow i)$. According to Theorem 1.4, because there exists a strict Lyapunov function V for system (1.1), its origin is asymptotically stable. The equation (2.3) implies, furthermore, that it is fixed-time stable. ■

Note that no assumptions on the regularity of $T(x)$ have been made, therefore, the conditions stated in Theorem 2.1 do not exclude the case of discontinuous $T(x)$. Also, the equation (2.2) is in general difficult to verify since it involves the explicit calculation of the trajectories ψ_x and of the inverse mapping θ. In what follows, more constructive conditions will be presented and the case of discontinuous $T(x)$ will be excluded.

Sufficient Conditions for FxTS with Continuous T

Theorem 2.2. *Suppose that there exists a continuously differentiable strict Lyapunov function* $V : \Gamma \to \mathbb{R}_{\geq 0}$ *for system* (1.1) *such that*

S1 *there exists a continuous positive definite function* $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *that verifies*

$$\int_0^{+\infty} \frac{dz}{r(z)} < +\infty;$$

S2 *the inequality* $\dot{V}(x) \leq -r(V(x))$ *holds for all* $x \in \Gamma$.

Then the origin of (1.1) *is fixed-time stable with continuous settling-time function* $T : \Gamma \to \mathbb{R}_{\geq 0}$ *and*

$$T(x) \leq \int_0^{+\infty} \frac{dz}{r(z)} \quad \forall x \in \Gamma. \tag{2.4}$$

Proof. Let us define the inverse mapping $(0, V(x)] \to [0, T(x))$ as $s \overset{\sigma}{\mapsto} t$, where $T : \Gamma \to \mathbb{R}_{\geq 0}$ is the settling-time function *i.e.* $T(x) := \inf\{T \geq 0 : V(\psi_x(T)) = 0\}$. Since V is, by assumption, differentiable on $x \in \Gamma$ and satisfies **S2**, $\sigma(s)$ is strictly decreasing and differentiable with

$$\sigma'(s) = \frac{1}{\dot{V}(\psi_x(\sigma(s)))}.$$

Then we have that for all $x \in \Gamma$

$$T(x) = \int_0^{T(x)} dt = -\int_{V(x)}^0 \frac{ds}{-\dot{V}(\psi_x(\sigma(s)))}.$$

From condition **S2** we have that $-\dot{V}(x) \geq r(V(x))$ for all $x \in \Gamma$ so that

$$\begin{aligned} -\int_{V(x)}^0 \frac{ds}{-\dot{V}(\psi_x(\sigma(s)))} &\leq \int_0^{V(x)} \frac{ds}{r(V(\psi_x(\sigma(s))))} \\ &\leq \int_0^{\sup_{x\in\Gamma} V(x)} \frac{ds}{r(V(\psi_x(\sigma(s))))} \\ &= \int_0^{+\infty} \frac{ds}{r(s)} < +\infty, \end{aligned}$$

where in this last step the condition **S1** was used. Therefore, $\sup_{x\in\Gamma} T(x) < +\infty$. Equivalently, the origin of (1.1) is fixed-time stable. Taking any $x_k \in \Gamma\backslash\{0\}$ converging to zero, we have that $T(x_k) \leq \int_0^{V(x_k)} \frac{ds}{r(s)}$ by continuity of V and since r was assumed to be positive definite we obtain

$$\lim_{x_k \to 0} \int_0^{V(x_k)} \frac{ds}{r(s)} = 0,$$

therefore T is continuous at the origin and due to Lemma 1.1.*ii*, T is continuous on Γ and, from the analysis above, bounded by (2.4). ∎

As can be seen from this theorem, fixed-time stability can be completely characterized by the pair of functions (V, r) and the regularity of this pair is linked to the regularity of the settling-time function. The following corollary gives more insight about some of the forms that the function r might take.

Corollary 2.1. *Suppose there exists a continuously differentiable strict Lyapunov function $V : \Gamma \to \mathbb{R}_{\geq 0}$ for system* (1.1) *and some constants $c_1, c_2 > 0$, $\alpha \in [0,1)$ and $\beta > 1$ such that*

$$\dot{V}(x) \leq -c_1 V(x)^{\alpha} - c_2 V(x)^{\beta} \quad \forall x \in \Gamma, \tag{2.5}$$

is satisfied. Then the origin of (1.1) *is fixed-time stable with continuous settling-time function T satisfying $T(x) \leq \frac{1}{c_1(1-\alpha)} + \frac{1}{c_2(\beta-1)}$.*

Proof. Although this result was proven in [Polyakov, 2012], in this proof we will use the characterization proposed in Theorem 2.2 and show that the results are consistent. Let us consider the ball $\mathcal{D} := \{x \in \Gamma : V(x) \leq 1\}$ and denote $T_{\mathcal{D}} := \inf\{x \in \Gamma \backslash \mathcal{D}, T \geq 0 : V(\psi_x(T)) \leq 1\}$, and $T_0 := \inf\{x \in \mathcal{D}, T \geq 0 : V(\psi_x(T)) = 0\}$ as the time functions that provide the time that takes to any trajectory outside $\mathcal{D}$ to arrive to $\mathcal{D}$, and the time that takes to any trajectory in $\mathcal{D}$ to arrive to zero, respectively. Then we have that for all $x \in \Gamma \backslash \mathcal{D}$, $\dot{V}(x) \leq -r_1(V(x))$, where $r_1(z) = c_2 z^{\beta}$, and therefore

$$\begin{aligned} T_{\mathcal{D}}(x) \leq \int_1^{\sup_{x\in\Gamma} V(x)} \frac{dz}{r_1(z)} &= \int_1^{+\infty} \frac{dz}{c_2 z^{\beta}} \\ &= \left. \frac{z^{1-\beta}}{c_2(\beta-1)} \right|_{+\infty}^{1} = \frac{1}{c_2(\beta-1)} < +\infty. \end{aligned}$$

For all $x \in \mathcal{D}$, we have that $\dot{V}(x) \leq -r_2(V(x))$, where $r_2(z) = c_1 z^{\alpha}$ and therefore

$$T_0(x) \leq \int_0^1 \frac{dz}{r_2(z)} = \int_0^1 \frac{dz}{c_1 z^{\alpha}} = \left. \frac{z^{1-\alpha}}{c_1(1-\alpha)} \right|_0^1 = \frac{1}{c_1(1-\alpha)} < +\infty.$$

Hence, **S1** and **S2** are satisfied with $r(z) = r_1(z) + r_2(z)$ and

$$T(x) = T_{\mathcal{D}} + T_0 \leq \frac{1}{c_1(1-\alpha)} + \frac{1}{c_2(\beta-1)} < +\infty \quad \forall x \in \Gamma.$$ ■

Example 2.2 ***Two dimensional systems***

Consider the systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -\lceil x_1 \rfloor^{\gamma} + x_2 \\ \dot{x}_2 = -\lceil x_2 \rfloor^{\gamma} - x_1 \end{cases}, \quad \Sigma_2 : \begin{cases} \dot{x}_1 = -\lceil x_1 \rfloor^{\gamma} - x_1^3 + x_2 \\ \dot{x}_2 = -\lceil x_2 \rfloor^{\gamma} - x_2^3 - x_1 \end{cases}, \quad x \in \mathbb{R}^2, \gamma \in (0,1)$$

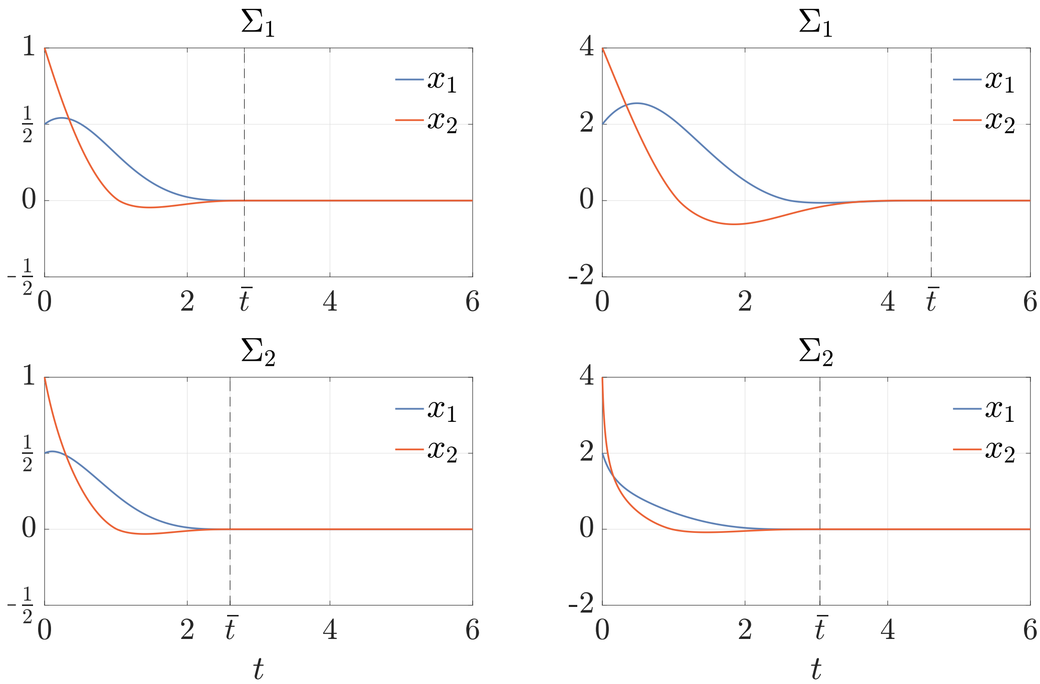

Figure 2.2 – Trajectories of Σ_1 and Σ_2 for $\gamma = \frac{1}{2}$ with initial conditions $x_0 = (-1, 2)$ (left) and $x_0 = (-5, 10)$ (right).

and the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. For Σ_1 we have

$$\dot{V}(x)\Big|_{\Sigma_1} = -(|x_1|^{\gamma+1} + |x_2|^{\gamma+1}) \leq -V(x)^{\frac{\gamma+1}{2}},$$

whereas for Σ_2

$$\dot{V}(x)\Big|_{\Sigma_2} = -(x_1^4 + x_2^4) - (|x_1|^{\gamma+1} + |x_2|^{\gamma+1}) \leq -V(x)^{\frac{\gamma+1}{2}} - V(x)^{\gamma+1},$$

where $\frac{\gamma+1}{2} < 1$ and $1 + \gamma > 1$. Then, according to Corollary 2.1 and Theorem 1.2, Σ_1 is finite-time stable and Σ_2 is fixed-time stable with $T(x) \leq \frac{1}{1-\gamma} + \frac{1}{\gamma}$. Figure 2.2 shows the trajectories of Σ_1 and Σ_2 under small initial conditions (left) and with slightly larger ones (right). The time $\bar{t}$ represents the instant at which $\|x\| \leq 10^{-3}$. It is possible to see how while the settling time of Σ_1 increases significantly with larger initial conditions, that of Σ_2 remains in a close vicinity.

2.2 Necessary Conditions for FxTS

In order to obtain the first result of this section, we will make use of the next lemma, whose proof can be found at the end of the chapter.

Lemma 2.1. *Suppose that the origin of system* (1.1) *is asymptotically stable on* Γ. *Then there exist a* *strict Lyapunov function* $\tilde{V}\colon \Gamma \to \mathbb{R}_{\geq 0}$ *for* (1.1)*, a continuous positive definite function* $\tilde{W}\colon \Gamma \to \mathbb{R}_{\geq 0}$ *and some* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *that satisfy*

M1 $\dot{\tilde{V}}(x) = -\tilde{W}(x) \quad \forall x \in \Gamma.$

M2 $\alpha_1(\tilde{V}(x)) \leq \tilde{W}(x) \leq \alpha_2(\tilde{V}(x)) \quad \forall x \in \Gamma.$

In fact, Lemma 2.1 is a corollary of Kurzweil's theorem (Theorem 1.4) and it states, in words, that if a given system is AS, then there exists a class-$\mathcal{K}_\infty$ function that satisfies, instead of the well known inequality $\dot{\tilde{V}}(x) < 0$, the equality **M1**.

Now we are ready to present a necessary condition for FxTS using a similar characterization as the one employed in Theorem 2.2.

Theorem 2.3. *Consider system* (1.1) *and suppose that the origin is fixed-time stable on* Γ*. Then there exist a strict Lyapunov function* V *and a class-*$\mathcal{K}_\infty$ *function* q *that verifies*

N1 $\displaystyle\int_0^{+\infty} \frac{dz}{q(z)} < +\infty;$

N2 $-q(V(x)) \leq \dot{V}(x) \qquad \forall x \in \Gamma.$

Proof. Since fixed-time stability implies asymptotic stability we know, from Lemma 2.1, that there exists a strict Lyapunov function V and a continuous positive definite function W that satisfy the **M** conditions. Then we have that for all $x \in \Gamma$:

I there exist a decreasing differentiable mapping $[0, T(x)] \to (0, V(x_0)]$, $t \mapsto V(\psi_x(t))$, with $T(x) = \inf\{T \geq 0 : V(\psi_x(T)) = 0\}$ and its corresponding inverse mapping $(0, V(x)] \to [0, T(x)]$, $s \overset{\sigma}{\mapsto} t$, also decreasing and differentiable such that $\sigma'(s) = 1/\dot{V}(\psi_x(\sigma(s)))$.

II Since $V(x)$ and $W(x) := \dot{V}(x)$ satisfy **M**, there exists some $q \in \mathcal{K}_\infty$ such that $-q(V(x)) \leq \dot{V}(x)\ \forall x \in \Gamma$ and **N2** is satisfied.

III Then

$$T(x) = \int_0^{T(x)} dt = -\int_{V(x)}^0 \frac{ds}{-\dot{V}(\psi_x(\sigma(s)))}.$$

From **II**, $-\dot{V}(x) \leq q(V(x))$ and therefore

$$-\int_{V(x)}^0 \frac{ds}{-\dot{V}(\psi_x(\sigma(s)))} \geq \int_0^{V(x)} \frac{ds}{q(V(\psi_x(\sigma(s))))}.$$

Hence

$$+\infty > \sup_{x \in \Gamma} T(x) \geq \int_0^{\sup_{x\in\Gamma} V(x)} \frac{ds}{q(s)}$$

and **N1** is fulfilled. ∎

It is not straightforward to obtain a converse result of Corollary 2.1 since some assumptions about the behavior of V when $x \to \partial\Gamma$ have to be made. However, for the case of uniform FxTS, the next necessary and sufficient condition can be obtained.

Theorem 2.4. *Consider the system* (1.1). *The following statements are equivalent:*

i) *The origin of* (1.1) *is uniformly fixed-time stable on* Γ *with continuous settling-time function* $T(x)$.

ii) *There exists a strict Lyapunov function* $V : \Gamma \to \mathbb{R}_{\geq 0}$ *and some positive constants* c_1, c_2 *and* c_3 *such that*

$$\dot{V}(x) = \begin{cases} -c_2 V(x)^2 & \text{if } V(x) \geq c_3 \\ -c_1 & \text{if } V(x) < c_3 \end{cases}. \tag{2.6}$$

Proof. $i) \Rightarrow ii)$. By assumption, the origin of (1.1) is fixed-time stable with continuous $T(x)$ and there exists some $T_m \in \mathbb{R}$ such that $\liminf_{\|x\|\to\partial\Gamma} T(x) = \limsup_{\|x\|\to\partial\Gamma} T(x) = T_m$ holds. Consider now the Lyapunov function candidate

$$V(x) := \begin{cases} \dfrac{1}{T_m - T(x)} & \text{if } T(x) \geq \frac{T_m}{2} \\ \frac{4}{T_m^2} T(x) & \text{if } T(x) < \frac{T_m}{2} \end{cases}.$$

Under the assumptions on $T(x)$, $V(x)$ is continuous, positive definite and radially unbounded for all $x \in \Gamma$. From Lemma 1.1.*i* we have that

$$\dot{T}(\psi_x(t)) = -1$$

and consequently

$$\dot{V}(x) = \begin{cases} -\left(\frac{1}{T_m - T(x)}\right)^2 & \text{if } T(x) \geq \frac{T_m}{2} \\ -\dfrac{4}{T_m^2} & \text{if } T(x) < \frac{T_m}{2} \end{cases} = \begin{cases} -V(x)^2 & \text{if } V(x) \geq \frac{2}{T_m} \\ -\dfrac{4}{T_m^2} & \text{if } V(x) < \frac{2}{T_m} \end{cases}.$$

Thus, (2.6) is satisfied with $c_1 = \frac{4}{T_m^2}$, $c_2 = 1$ and $c_3 = \frac{2}{T_m}$.
$ii) \Rightarrow i)$. From previous arguments, we have that for all $x \in \Gamma$ such that $V(x) \geq c_3$, the chain of equalities

$$T(x) = \int_{V(x)}^{c_3} \frac{ds}{\dot{V}(\psi_x(\theta(s)))} + \int_{c_3}^{0} \frac{ds}{\dot{V}(\psi_x(\theta(s)))} = \int_{c_3}^{V(x)} \frac{ds}{c_2 s^2} + \int_0^{c_3} \frac{ds}{c_1}$$

holds and

$$T(x) = \frac{1}{c_2 c_3} - \frac{1}{c_2 V(x)} + \frac{c_3}{c_1}.$$

Since $V(x)$ is continuous, $T(x)$ is also continuous. Moreover, since $V(x)$ is radially unbounded, it follows that

$$\liminf_{x\to\partial\Gamma} T(x) = \limsup_{x\to\partial\Gamma} T(x) = \frac{1}{c_2 c_3} + \frac{c_3}{c_1}.$$

For all $x \in \Gamma$ such that $V(x) < c_3$, the settling-time function gets reduced to

$$T(x) = \frac{V(x)}{c_1} < \frac{c_3}{c_1}.$$

Gathering this arguments we conclude that the origin of (1.1) is uniformly fixed-time stable with $T_m = \frac{1}{c_2 c_3} + \frac{c_3}{c_1}$. ■

2.3 FxT Stabilization of Nonlinear Affine Systems

In this section, we will give a sufficient conditions for fixed-time stabilization following a similar structure of well known results on asymptotic stabilization of autonomous systems.

Consider the following affine in the input u system:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i, \quad x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m, \tag{2.7}$$

where $f_0(0) = 0$, f_i is continuous for all $0 \leq i \leq m$ and such that (2.7) has uniqueness of solutions in forward time. Its closed-loop representation is given by

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i(x), \quad x \in \mathbb{R}^n. \tag{2.8}$$

Let us recall the definition of *stabilization* and propose a definition of *fixed-time stabilization*. In the latter, we will only consider fixed-time stabilization with continuous settling-time functions.

Definition 2.2. The control system (2.7) is *stabilizable* (respectively *fixed-time stabilizable*) if there exists a nonempty neighborhood of the origin $\Gamma \subseteq \mathbb{R}^n$ and a $\mathcal{C}^0$ feedback control law $u : \Gamma \to \mathbb{R}^m$ such that:

1. $u(0) = 0$;
2. the origin of the system (2.8) is asymptotically stable (respectively fixed-time stable with a continuous settling-time function).

Such a feedback law $u(x)$ is called a *stabilizer* (respectively *fixed-time stabilizer*) for system (2.7).

A radially unbounded, positive definite, $\mathcal{C}^1$ function $V : \Gamma \to \mathbb{R}_{\geq 0}$ is a *control Lyapunov function* (CLF) for the system (2.7) if for all $x \in \Gamma \backslash \{0\}$,

$$\inf_{u \in \mathbb{R}^m} (a(x) + \langle B(x), u \rangle) < 0, \tag{2.9}$$

where $a(x) = \mathcal{L}_{f_0} V(x)$, $B(x) = (B_1(x), \ldots, B_m(x))$ with $B_i(x) = \mathcal{L}_{f_i} V(x)$ for $1 \leq i \leq m$. Such a control Lyapunov function satisfies the *small control property* (SCP) if for each $\epsilon > 0$, there exists $\tau > 0$ such that, if $x \in \tau \mathcal{B}^n$, then there exists some $u \in \epsilon \mathcal{B}^m$ such that

$$a(x) + \langle B(x), u \rangle < 0. \tag{2.10}$$

For a CLF, $a(x) < 0$ whenever $B(x) = 0$, and one may think that this is a necessary condition for asymptotic stabilization. However, as shown in [Sontag, 1989], it is also a sufficient one.

Indeed, E. Sontag shows that if a radially unbounded and positive definite $\mathcal{C}^1$ function V satisfies $b(x) = 0 \Rightarrow a(x) < 0$ where $b(x) = \|B(x)\|^2$, then the feedback law $u = w(x)$,

$$w_i(x) := \begin{cases} -\dfrac{a(x) + \sqrt{a(x)^2 + b(x)^2}}{b(x)} B_i(x) & \text{if } x \in \Gamma \backslash \{0\} \\ 0 & \text{if } x = 0 \end{cases}, \tag{2.11}$$

known as *Sontag's universal formula*, is a stabilizer for (2.7). It was also shown in [Sontag, 1989] that this feedback law is continuous on $\Gamma\backslash\{0\}$ and that if V additionally satisfies the SCP, then w is continuous on Γ (see, for instance, [Isidori, 2013, Chapter 9]).

The following theorem presents an analogous formulation for fixed-time stabilization and in order to prove it, we will provide a continuous fixed-time stabilizer, akin to Sontag's universal formula.

Theorem 2.5. *Consider the system described by* (2.7)*. There exists a continuous fixed-time stabilizer* $u = v(x)$ *for* (2.7) *if there exists a radially unbounded and positive definite* $\mathcal{C}^1$ *control Lyapunov function* $V : \Gamma \to \mathbb{R}_{\geq 0}$ *that satisfies the small control property and*

$$a(x)^2 + b(x)^2 \rho(x)^2 \geq (c_1 V(x)^\alpha + c_2 V(x)^\beta)^2 \quad \forall x \in \Gamma, \tag{2.12}$$

for some $c_1, c_2 > 0$*,* $\alpha \in (0,1)$ *and* $\beta > 1$*, where* $\rho : \Gamma \to \mathbb{R}_{\geq 0}$ *is a continuous function on* $\Gamma\backslash\{0\}$ *such that*

$$\limsup_{x \to 0} \rho(x)\sqrt{b(x)} < +\infty. \tag{2.13}$$

Proof. **I** Let us introduce, for brevity in the notation, the function $\varphi(s) := c_1 s^\alpha + c_2 s^\beta$ and define the feedback law

$$v_i(x) := \begin{cases} -\dfrac{a(x) + \sqrt{a(x)^2 + b(x)^2 \tilde{\rho}(x)^2}}{b(x)} B_i(x) & \text{if } x \in \Gamma \backslash \{0\} \\ 0 & \text{if } x = 0 \end{cases}, \tag{2.14}$$

where $\tilde{\rho}(s) := (1 + \rho(x))$. The derivative of V along the trajectories of (2.7)-(2.14) is given by

$$\dot{V}(x) = \left\langle \nabla V(x), f_0(x) + \sum_{i=1}^m f_i(x) v_i(x) \right\rangle = -\sqrt{a(x)^2 + b(x)^2 \tilde{\rho}(x)^2} < 0 \quad \forall x \in \Gamma\backslash\{0\}. \tag{2.15}$$

Thus the control (2.14) is a stabilizer for (2.7) and V is a strict Lyapunov function for the closed-loop system (2.8).

II Since $\tilde{\rho}(x) \geq 1$ for all $x \neq 0$, using the change of variables $\tilde{B}_i(x) = B_i(x)\sqrt{\tilde{\rho}(x)}$, $\tilde{b} = b(x)\tilde{\rho}(x)$ in (2.14) we obtain

$$v_i(x) = \begin{cases} -\sqrt{1 + \rho(x)} \dfrac{a(x) + \sqrt{a(x)^2 + \tilde{b}(x)^2}}{\tilde{b}(x)} \tilde{B}_i(x) & \text{if } x \in \Gamma \backslash \{0\} \\ 0 & \text{if } x = 0 \end{cases},$$

so that Sontag's universal formula is recovered multiplied by a gain $\sqrt{1+\rho(x)}$, which is continuous for all $x \in \Gamma \backslash \{0\}$. Then, since by assumption V satisfies the SCP and $\rho(x)\sqrt{b(x)}$ is bounded at the origin, (2.14) is a continuous feedback stabilizer for (2.7).

III Now let us show that the control (2.14) is a fixed-time stabilizer for (2.7). We have from (2.15) that

$$\begin{aligned}
\dot{V}(x) &= -\left(\left(\frac{a(x)}{\varphi(V(x))}\right)^2 + \left(\frac{(1+\rho(x))b(x)}{\varphi(V(x))}\right)^2\right)^{\frac{1}{2}} \varphi(V(x)) \\
&\leq -\left(\frac{a(x)^2 + \rho(x)^2 b(x)^2}{\varphi(V(x))^2}\right)^{\frac{1}{2}} \varphi(V(x)) \\
&\leq -\varphi(V(x)),
\end{aligned}$$

which implies fixed-time stability for all $x \in \Gamma$ and from Corollary 2.1, (2.14) is moreover a continuous fixed-time stabilizer for (2.7). ■

Example 2.3

Consider the following system

$$\begin{aligned}
\dot{x}_1 &= -\lceil x_1 \rfloor^{\mu} - \lceil x_1 \rfloor^{1+\mu} - x_2 \\
\dot{x}_2 &= \lceil x_1 \rfloor^{\eta} |x_2|^{1-\eta} + |x_2|^{\frac{\mu-\eta}{2}} u
\end{aligned} \tag{2.16}$$

where $\eta \in (0,1), \mu \in (0,1)$ and $\mu > \eta$. By choosing the $\mathcal{C}^1$ control Lyapunov function candidate $V(x) = \frac{1}{\eta+1}|x_1|^{\eta+1} + \frac{1}{\eta+1}|x_2|^{\eta+1}$, we have that

$$\begin{aligned}
a(x) &= -|x_1|^{\eta+\mu} - |x_1|^{1+\eta+\mu}, \\
B(x) &= |x_2|^{\frac{\eta+\mu}{2}}, \quad b(x) = |x_2|^{\eta+\mu}.
\end{aligned}$$

Since $\inf_{u \in \mathbb{R}}(a(x) + Bu) < 0$ for $x \neq 0$, V is a CLF for the system. Selecting $\rho(x) = 1 + |x_2|$ we arrive to

$$|a(x)| + b(x)\rho(x) = |x_1|^{\eta+\mu} + |x_2|^{\eta+\mu} + |x_1|^{1+\eta+\mu} + |x_2|^{1+\eta+\mu}.$$

From Jensen's inequality and Lemma 2.3 of [Qian and Lin, 2001] we obtain for any $p \geq 1$ and all $z_1, z_2 \in \mathbb{R}$:

$$\begin{aligned}
|z_1|^p + |z_2|^p &\geq 2^{1-p}(|z_1| + |z_2|)^p, \\
(|z_1| + |z_2|)^{1/p} &\leq |z_1|^{1/p} + |z_2|^{1/p},
\end{aligned}$$

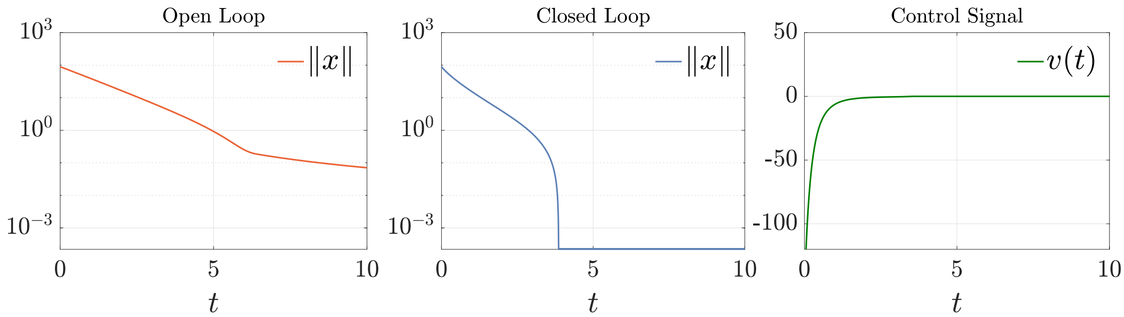

Figure 2.3 – State norm $\|x\|$ in logarithmic scale of the system (2.16) in open loop $u = 0$ (left), in closed loop $u = v(x)$ (middle) and the control signal $v(t)$ (right) for $x_0 = (-40, 80)$, $\mu = 1/4$ and $\eta = 1/5$.

which in our case can be used as follows:

$$|x_1|^{\eta+\mu} + |x_2|^{\eta+\mu} = \left(|x_1|^{\eta+1}\right)^{\frac{\eta+\mu}{\eta+1}} + \left(|x_2|^{\eta+1}\right)^{\frac{\eta+\mu}{\eta+1}} \geq \left(|x_1|^{\eta+1} + |x_2|^{\eta+1}\right)^{\frac{\eta+\mu}{\eta+1}},$$
$$|x_1|^{\eta+\mu+1} + |x_2|^{\eta+\mu+1} = \left(|x_1|^{\eta+1}\right)^{\frac{\eta+\mu+1}{\eta+1}} + \left(|x_2|^{\eta+1}\right)^{\frac{\eta+\mu+1}{\eta+1}} \geq 2^{-\frac{\mu}{\eta+1}} \left(|x_1|^{\eta+1} + |x_2|^{\eta+1}\right)^{\frac{\eta+\mu+1}{\eta+1}},$$

and

$$|a(x)| + b(x)\rho(x) \geq (\eta+1)^{\frac{\eta+\mu}{\eta+1}} V(x)^{\frac{\eta+\mu}{\eta+1}} + 2^{-\frac{\mu}{\eta+1}} (\eta+1)^{\frac{\eta+\mu+1}{\eta+1}} V(x)^{\frac{\eta+\mu+1}{\eta+1}}.$$

Since the inequalities above hold, then

$$(|a(x)| + b(x)\rho(x))^2 \geq \varphi(V(x))^2 \tag{2.17}$$

where $\varphi(V(x)) = (\eta+1)^{\delta_1} V(x)^{\delta_1} + 2^{1-\delta_2}(\eta+1)^{\delta_2} V(x)^{\delta_2}$, $\delta_1 = \frac{\eta+\mu}{\eta+1} < 1$ and $\delta_2 = \frac{1+\eta+\mu}{\eta+1} > 1$, and the condition (2.12) is fulfilled. It can be easily checked that (2.13) is satisfied and since $\frac{a(x)}{|B(x)|} \leq 0$, V satisfies the small control property. Then all the conditions of Theorem 2.5 are met and the feedback law

$$u = v(x) = \frac{|x_1|^{\eta+\mu} + |x_1|^{1+\eta+\mu} - \sqrt{\left(|x_1|^{\eta+\mu} + |x_1|^{1+\eta+\mu}\right)^2 + (2+|x_2|)^2 |x_2|^{\eta+\mu}}}{|x_2|^{\frac{\eta+\mu}{2}}}$$

is a continuous fixed-time stabilizer for (2.16). Figure 2.3 shows, from left to right, the norm of the state for the open loop system (2.16) *i.e.* $u = 0$, the closed loop system with $u = v(x)$ and the control signal. It is posible to see that although the open loop system is stable, its convergence rate is not in fixed-time. The middle plot in Figure 2.3 shows that the controller $v(x)$ enforces fixed-time stability and in the right-hand plot we can see that the control signal $v(t)$ is indeed continuous.

2.4 Conclusions

Complete necessary and sufficient conditions for fixed-time stability of continuous autonomous systems have been presented. A characterization of this property using a pair of functions has been proposed and it allows, in the sufficiency case, to rule out the case of discontinuous settling-time function. It is worth noticing that in the sufficiency case, the characterizing function r is continuous and positive and this assumption is enough to assert fixed-time stability with continuous T. In the necessary case, however, the characterizing function q is not only continuous and positive definite but also increasing and unbounded *i.e.* a class-$\mathcal{K}_\infty$ function and no assumptions on the regularity of T were made. A particular, more constructive and previously studied form of the characterizing function r has shown to be consistent with the framework here presented.

The concept of uniform fixed-time stability has been introduced and a necessary and sufficient condition for this property has been obtained. Finally, a sufficient condition for fixed-time stabilization of continuous affine systems, analogous to previous results on asymptotic stabilization, has also been obtained. It is left as an open problem, to find a necessary condition for fixed-time stabilization of affine systems.

2.5 Proofs

Proof of Corollary 2.1. Following Theorem 1.4, there exists a continuous positive definite function V such that

$$W^*(x) := -\frac{\partial V}{\partial x} f(x) > 0 \quad \forall x \in \Gamma \backslash \{0\}.$$

Let us propose the function $\widetilde{V}(x) = \int_0^{V(x)} \xi(p)dp$, where $\xi \in \mathcal{K}_\infty$ is defined later on. Note that from its definition, $\widetilde{V}$ is continuous and radially unbounded. Then

$$\dot{\widetilde{V}}(x) = \xi(V(x))\dot{V}(x) = -\xi(V(x))W^*(x).$$

By defining $\widetilde{W}(x) := \xi(V(x))W^*(x)$ we obtain **M1** and it becomes clear that $\widetilde{W}$ is continuous, $\widetilde{W}(0) = 0$ and $\widetilde{W}(x) > 0$ for all $x \in \Gamma \backslash \{0\}$ such that **M2** is satisfied with a suitably selected ξ and the class-$\mathcal{K}_\infty$ functions

$$\alpha_1(s) = \frac{s}{s+1} \inf_{x \in \Gamma : \widetilde{V}(x) \geq s} \widetilde{W}(x), \qquad \alpha_2(s) = s + \sup_{x \in \Gamma : \widetilde{V}(x) \leq s} \widetilde{W}(x).$$

Since $\widetilde{V}$ satisfies all conditions of Definition 1.2, it constitutes a strict Lyapunov function for system (1.1). ∎

Chapter 3

NonA ISS Lyapunov Functions

Contents

This chapter gathers the notions of nonasymptotic stability rates with input-to-state stability. We will start by defining the properties of finite-time ISS and fixed-time ISS followed by its explicit Lyapunov characterization. Through academic examples and simulations we will compare asymptotic ISS versus NonA ISS and we will show that NonA ISS might be preferable, both in terms of robustness and convergence rate. In the last section we will use the results of the explicit framework to develop the implicit one. The chapter closes with an example that illustrates the capabilities of the implicit NonA ISS framework.

It is worth to mention that Y. Hong, Z.P. Jiang and G. Feng have worked extensively in the topic of finite-time ISS [Hong et al., 2010]. Many of the results to be presented here incorporate the notions developed in Hong's work and extend them to the fixed-time case using both the explicit and the implicit approach.

Let us recall the system under study:

$$\dot{x} = f(x,d), \quad t \geq 0, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the input signal and the same assumptions made in Section 1.2 about f and d hold.

3.1 Nonasymptotic Input-to-State Stability

The definitions of NonA ISS will make use of a special kind of comparison functions, later in the chapter it will become clear why class-$\mathcal{KL}$ functions are not suitable to describe this property, as it was the case for asymptotic ISS (see Theorem 1.3).

Definition 3.1. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *generalized class-$\mathcal{KL}$ function* ($\mathcal{GKL}$ function) if

i) the mapping $s \mapsto \beta(s, 0)$ is a class-$\mathcal{K}$ function;

ii) for each fixed $s \geq 0$ the mapping $t \mapsto \beta(s,t)$ is continuous, decreases to zero and there exists some $\widetilde{T}(s) \in [0, +\infty)$ such that $\beta(s,t) = 0$ for all $t \geq \widetilde{T}(s)$.

Compared to $\mathcal{KL}$ functions, a $\mathcal{GKL}$ function has to be a $\mathcal{K}$ function only for $t = 0$ whereas a $\mathcal{KL}$ function has to be so for any fixed $t \geq 0$. Moreover, a $\mathcal{GKL}$ function not only has to be continuous and decreasing for each fixed s, but also has to converge to zero in a finite time[1].

Definition 3.2. The system (3.1) is said to be *finite-time ISS* (FTISS) if for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty$, each solution $\psi_{x_0}(t,d)$ is defined for $t \geq 0$ and satisfies

$$\|\psi_{x_0}(t,d)\| \leq \beta(\|x_0\|, t) + \vartheta(\|d\|_{[0,\infty)}) \quad \forall t \geq 0, \tag{3.2}$$

where ϑ is a class-$\mathcal{K}$ function and β is a class-$\mathcal{GKL}$ function with $\beta(r,t) = 0$ when $t \geq \widetilde{T}(r)$ with $\widetilde{T}(r)$ continuous with respect to r and $\widetilde{T}(0) = 0$. The system (3.1) is said to be *fixed-time ISS* (FXISS) if it is FTISS and $\sup_{r \in \mathbb{R}_{\geq 0}} \widetilde{T}(r) < +\infty$.

Remark that indeed the key difference with respect to asymptotic ISS is that β is a $\mathcal{GKL}$ function and that according to Definition 3.1 this implies the existence of the settling-time function $\widetilde{T}$. Remark also that only the case of continuous $\widetilde{T}$ is considered.

Example 3.1 (Hong et al., 2010)

The trajectories of the input scalar system

$$\dot{x} = -\lceil x \rfloor^{\frac{1}{3}} - x^3 + d^2 \tag{3.3}$$

satisfy, for all $x, d \in \mathbb{R}$, the inequality (3.2) with

$$\beta(s,t) = \begin{cases} (s^{\frac{2}{3}} - \frac{1}{3}t)^{\frac{3}{2}} & \text{if } 0 \leq t \leq \widetilde{T}(s) \\ 0 & \text{otherwise} \end{cases}, \quad \widetilde{T}(s) = 3s^{\frac{2}{3}}, \quad \gamma(s) = 2s^2.$$

Since $\beta(s,0) = s$ is a class-$\mathcal{K}$ function, $\beta(s,t) = 0$ for all $t \geq \widetilde{T}(s)$ and for each fixed s, $\beta(s,t)$ is decreasing (see Figure 3.1), β is a class-$\mathcal{GKL}$ function and the system (3.3) is FTISS. Figure

[1] Note that the definition of a $\mathcal{GKL}$ function presented here differs from the one introduced in [Hong et al., 2010].

3.1 also shows, in blue, the projection of $\beta(s,0)$ on the s-β axis. The white line on the t-s axis represents the settling-time curve $t = 3s^{\frac{2}{3}}$.

The plot of β in Figure 3.1 also helps to understand the use of class-$\mathcal{GKL}$ functions: by looking at the grid lines on the s axis, it is possible to notice that for any fixed $t \neq 0$, $\beta(s,t)$ is not a class-$\mathcal{K}$ function since it equals to zero for multiple values of s. Hence, the FTISS property cannot be defined using $\mathcal{KL}$ functions.

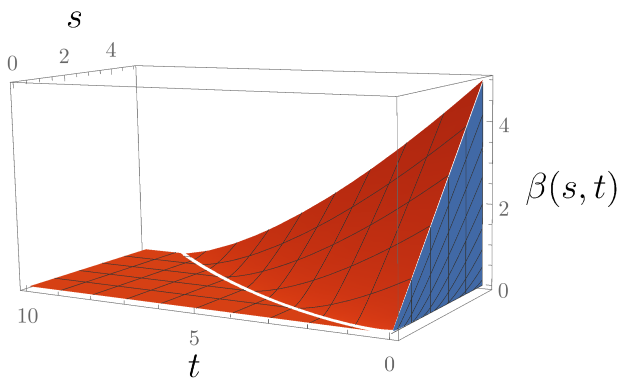

Figure 3.1 – 3D plot of the function $\beta(s,t)$ of Example 3.1.

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From the definition of FTISS, it follows that if system (3.1) is FTISS, for $d = 0$ it becomes FTS with some continuous settling-time function T. Therefore, for FTISS systems, the existence of $T(x)$ implies that of $\tilde{T}(x)$ and vice versa. Hence FTISS implies FTS when $d = 0$, however the converse is in general not true.

Example 3.2

The state of the system

$$\dot{x} = -(1 + \sin d)\lceil x \rfloor^{\frac{1}{3}} \tag{3.4}$$

is bounded for each bounded input $d \in \mathbb{R}$. Moreover, for $d = 0$ the origin of (3.4) is FTS. However, for $d = 3\pi/2$, the origin is not even AS and therefore not FTISS.

3.2 Explicit Characterization

Definition 3.3. Consider a positive definite and radially unbounded $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. V is called a *finite-time ISS Lyapunov function* for system (3.1) if there exist some $\chi \in \mathcal{K}$, $c > 0$ and $\alpha \in [0,1)$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -cV(x)^{\alpha}. \tag{3.5}$$

V is called a *fixed-time ISS Lyapunov function* for system (3.1) if there exist some $\chi \in \mathcal{K}$, $c_1, c_2 > 0$,

$\alpha \in [0,1)$ and $\beta > 1$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -c_1 V(x)^{\alpha} - c_2 V(x)^{\beta} \tag{3.6}$$

In [Hong et al., 2010], some sufficient conditions for finite-time ISS with continuous settling-time function are presented. However converse results are not obtained. The following result shows that if some assumptions about Lipschitz continuity of the system and of the settling-time function are added, then a converse result can be obtained.

Assumption 3.1. *Let on any compact set $K \subset (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^m$ the function $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ in* (3.1) *be Lipschitz and, in addition, suppose there exists some continuous function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\|f(x,d) - f(x,0)\| \leq L(\|x\|)\|d\|$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.

Theorem 3.1. *The system* (3.1) *is FTISS if it admits a finite-time ISS Lyapunov function. Conversely, if* (3.1) *is FTISS with a Lipschitz continuous settling-time function $\widetilde{T}$ and Assumption 3.1 is satisfied, then there exists a finite-time ISS Lyapunov function for it.*

As it was the case in Chapter 2, in order to obtain a complete characterization, *i.e.* necessary and sufficient conditions for FXISS some assumptions on the uniformity of the settling-time function with respect to x are required. Therefore we will present only a sufficient Lyapunov condition for FXISS, however, using the notions of uniform fixed-time stability, introduced in Chapter 2, and the conditions of Theorem 2.4, a converse result on FXISS can be obtained.

Theorem 3.2. *The system* (3.1) *is FXISS if it admits a fixed-time ISS Lyapunov function.*

Naturally, whenever $d = 0$ and the above theorems' conditions are fulfilled, the corresponding settling-time estimates (1.15) and (1.16) hold.

With the Lyapunov tools for FTISS and FXISS now at hand, let us illustrate in more detail some of the differences with respect to asymptotic ISS.

Example 3.3 ***ISS vs FTISS***

Consider the input systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -x_1 - x_2 + d_1 \\ \dot{x}_2 = x_1 - x_2^3 + d_2 \end{cases}, \quad \Sigma_2 : \begin{cases} \dot{x}_1 = -\lceil x_1 \rfloor^{\gamma} - x_2 + d_1 \\ \dot{x}_2 = x_1 - \lceil x_2 \rfloor^{\gamma} + d_2 \end{cases}, \quad x, d \in \mathbb{R}^2, \gamma \in (0,1)$$

and the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

For the system Σ_1 we have

$$\begin{aligned}\dot{V}\Big|_{\Sigma_1} &= -x_1^2 - x_2^4 + x_1 d_1 + x_2 d_2 \\ &\leq -(1-\theta)x_1^2 - (1-\theta)x_2^4 - \theta x_1^2 - \theta x_2^4 + |x_1|\,\|d\|_\infty + |x_2|\,\|d\|_\infty\end{aligned}$$

where $0 < \theta < 1$ and $\|d\|_\infty = \max\{|d_1|, |d_2|\}$. Next we have that

$$\dot{V}\Big|_{\Sigma_1} \leq -(1-\theta)x_1^2 - (1-\theta)x_2^4, \quad \text{for } |x_1| \geq \frac{\|d\|_\infty}{\theta} \text{ and } |x_2| \geq \left(\frac{\|d\|_\infty}{\theta}\right)^{1/3}.$$

For $|x_2| \leq (\|d\|_\infty/\theta)^{1/3}$ we obtain

$$\begin{aligned}\dot{V}\Big|_{\Sigma_1} &\leq -x_1^2 - x_2^4 + |x_1|\,\|d\|_\infty + \frac{(\|d\|_\infty)^{4/3}}{\theta^{1/3}} \\ &= -(1-\theta)x_1^2 - x_2^4 - \theta x_1^2 + |x_1|\,\|d\|_\infty + \frac{(\|d\|_\infty)^{4/3}}{\theta^{1/3}}.\end{aligned}$$

Let $\rho_1(r)$ be the largest positive real root of the quadratic equation

$$-\theta y^2 + ry + \frac{r^{4/3}}{\theta^{1/3}} = 0, \quad r \geq 0.$$

Then ρ_1 is a class-$\mathcal{K}_\infty$ function and $\rho_1(r) \geq r/\theta$. Hence, for $|x_2| \leq (\|d\|_\infty/\theta)^{1/3}$ we have

$$\dot{V}\Big|_{\Sigma_1} \leq -(1-\theta)x_1^2 - x_2^4, \quad \text{for } |x_1| \geq \rho_1(\|d\|_\infty).$$

Proceeding similarly for the case $|x_1| \leq \|d\|_\infty/\theta$, we arrive to

$$\dot{V}\Big|_{\Sigma_1} \leq -x_1^2 - (1-\theta)x_2^4, \quad \text{for } |x_2| \geq \rho_2(\|d\|_\infty),$$

where $\rho_2(r)$ is the largest positive real root of the quartic equation

$$-\theta y^4 + ry + \frac{r^2}{\theta} = 0, \quad r \geq 0.$$

Gathering the above estimates and defining $\rho_3(r) = \max\{\rho_1(r), \rho_2(r)\}$ we obtain

$$\dot{V}\Big|_{\Sigma_1} \leq -(1-\theta)x_1^2 - (1-\theta)x_2^4, \quad \forall\ \|x\|_\infty \geq \rho_3(\|d\|_\infty).$$

From Theorem 1.7 we conclude that Σ_1 is an ISS system.
Using a similar analysis for Σ_2 it is not difficult to show that

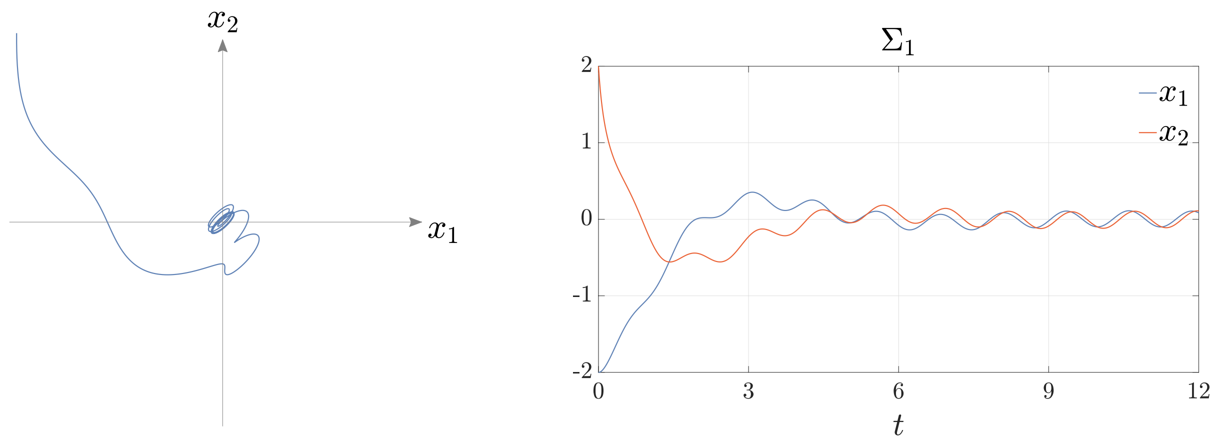

Figure 3.2 – Phase space diagram (left) and time plot (right) of a trajectory of the ISS system Σ_1 with initial condition $x_0 = (-2, 2)$ and $d_1(t) = d_2(t) = 0.5\sin(5t)$.

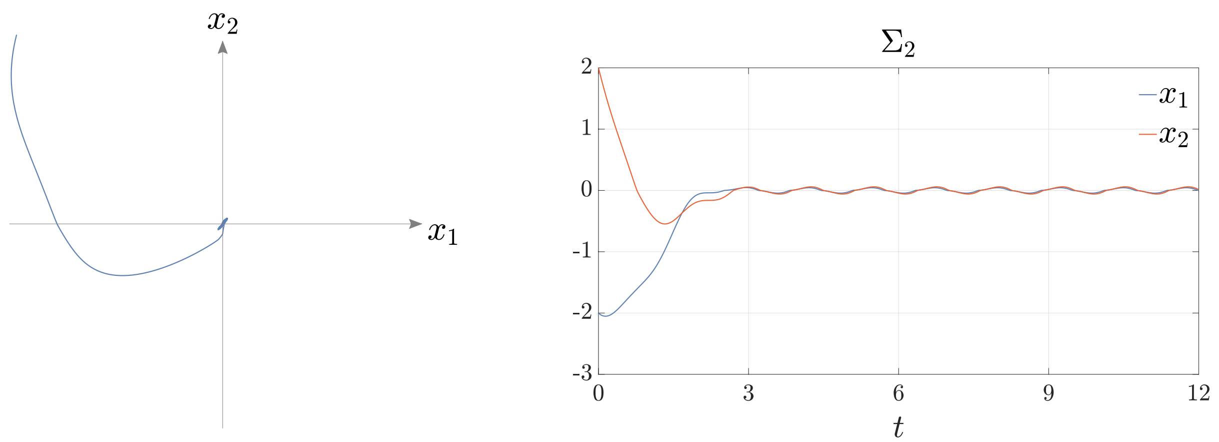

Figure 3.3 – Phase space diagram (left) and time plot (right) of a trajectory of the FTISS system Σ_2 with initial condition $x_0 = (-2, 2)$ and $d_1(t) = d_2(t) = 0.5\sin(5t)$.

$$\dot{V}\Big|_{\Sigma_2} \leq \begin{cases} -(1-\theta)(|x_1|^{\gamma+1} + |x_2|^{\gamma+1}), & \text{for } |x_1| \geq \left(\frac{\|d\|_\infty}{\theta}\right)^{\frac{1}{\gamma}} \text{ and } |x_2| \geq \left(\frac{\|d\|_\infty}{\theta}\right)^{\frac{1}{\gamma}}. \\ -(1-\theta)|x_1|^{\gamma+1} - |x_2|^{\gamma+1}, & \text{for } |x_1| \geq \frac{2^{\gamma+1}}{\theta^{1/\gamma}}\|d\|_\infty^{\frac{1}{\gamma}} \text{ and } |x_2| \leq \left(\frac{\|d\|_\infty}{\theta}\right)^{\frac{1}{\gamma}}. \\ -|x_1|^{\gamma+1} - (1-\theta)|x_2|^{\gamma+1}, & \text{for } |x_1| \leq \left(\frac{\|d\|_\infty}{\theta}\right)^{\frac{1}{\gamma}} \text{ and } |x_2| \geq \frac{2^{\gamma+1}}{\theta^{1/\gamma}}\|d\|_\infty^{\frac{1}{\gamma}}. \end{cases}$$

Thus

$$\|x\|_\infty \geq \rho_4(\|d\|_\infty) \Rightarrow \dot{V}\Big|_{\Sigma_2} \leq -(1-\theta)V(x)^{\frac{\gamma+1}{2}},$$

where $\rho_4(r) = \frac{2^{\gamma+1}}{\theta^{1/\gamma}} r^{\frac{1}{\gamma}}$ and $\frac{\gamma+1}{2} < 1$. Then, according to Theorem 3.1, Σ_2 is an FTISS system. Figure 3.2 shows a trajectory of Σ_1 starting at $x_0 = (-2, 2)$ with inputs $d_1(r) = d_2(t) = 0.5\sin(5t)$. Figure 3.3 shows a trajectory of Σ_2 starting at the same initial condition and with the same inputs. It becomes noticeable that the trajectories of Σ_2 are less influenced by the disturbance d and that they remain contained in a smaller vicinity of the origin.

3.3 Implicit Characterization

Let us begin this section with the implicit characterization of asymptotic ISS and by recalling that the set $\Omega \subset \mathbb{R}^{n+1}$ was defined in Theorem 1.13 as $\Omega := \left\{(V,x) \in \mathbb{R}^{n+1} : Q(V,x) = 0\right\}$.

Definition 3.4. A continuous function $Q : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ is called an *implicit ISS Lyapunov function* for system (3.1) if it satisfies the conditions **C1**-**C4** of Theorem 1.13 for $d = 0$ and

C5$^{\text{iss}}$ $\quad \|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \gamma(\|x\|)$

for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$, with $\chi, \gamma \in \mathcal{K}$.

Theorem 3.3. *System* (3.1) *is ISS if and only if there exists an implicit ISS Lyapunov function* $Q(V,x)$ *for it.*

As in the explicit case, the implicit formulation can also benefit from an alternative definition, formulated in dissipativity-like terms.

Corollary 3.1. *Suppose that there exists a continuous function* $Q : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ *that satisfies the conditions* **C1**-**C4** *of Theorem 1.13 for* $d = 0$ *and*

C5$^{\text{iss*}}$ $\quad \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \zeta(\|x\|) - \delta(\|d\|)$

for all $(V,x) \in \Omega$ *and all* $d \in \mathbb{R}^m$*, where* $\delta, \zeta \in \mathcal{K}_\infty$*. Then* (3.1) *is ISS and* $Q(V,x)$ *is an implicit ISS Lyapunov function for* (3.1).

Implicit NonA ISS Lyapunov Functions

Definition 3.5. Consider a continuous function $Q : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$ that satisfies all conditions of Theorem 1.13 for $d = 0$. Q is called an *implicit finite-time ISS Lyapunov function* for (3.1) if there exist some $\chi \in \mathcal{K}$, $c_1 > 0$ and $\alpha \in (0,1)$ such that for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$

C5$^{\text{ft}}$ $\quad \|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c V^\alpha.$

Q is called an *implicit fixed-time ISS Lyapunov Function* for (3.1) if there exist some $\chi \in \mathcal{K}$, $c_1, c_2 > 0$, $\alpha \in [0,1)$ and $\beta > 1$ such that for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$

C5$^{\text{fx}}$ $\quad \|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^\alpha + c_2 V^\beta.$

An alternative Lyapunov characterization of FXISS can also be established by using two functions $Q_1(V,x)$ and $Q_2(V,x)$ that define, implicitly and in a piecewise fashion, the function V. This characterization is analogous to the one made in Theorem 1.15.

Theorem 3.4. *System* (3.1) *is FTISS if there exists an implicit finite-time ISS Lyapunov function for it. Conversely, if* (3.1) *is FTISS with a Lipschitz continuous settling-time function* $\widetilde{T}$ *and Assumption 3.1 holds, then there exists an implicit finite-time Lyapunov function for it.*

Theorem 3.5. *System* (3.1) *is FXISS if it admits an implicit fixed-time ISS Lyapunov function.*

Clearly, the implicit approach inherits from the explicit one both the dissipativity-like characterization and the settling-time estimates (1.15) and (1.16), provided that $d = 0$.

Corollary 3.2. *Consider a continuous function* $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *that satisfies conditions* **C1**-**C4** *for* $d = 0$. *If there exist some* $\delta \in \mathcal{K}_\infty$, $c > 0$ *and* $\alpha \in (0,1)$ *such that*

$$\mathbf{C5}^{\mathrm{ft}^*} \quad \frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq cV^\alpha - \delta(\|d\|)$$

for all $(V,x) \in \Omega$ *and all* $d \in \mathbb{R}^m$, *then* (3.1) *is FTISS and* $Q(V,x)$ *is an implicit finite-time ISS Lyapunov function for* (3.1). *If there exist some* $\delta \in \mathcal{K}_\infty$, $c_1, c_2 > 0$, $\alpha \in (0,1)$ *and* $\beta > 1$ *such that*

$$\mathbf{C5}^{\mathrm{fx}^*} \quad \frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^\alpha + c_2 V^\beta - \delta(\|d\|)$$

for all $(V,x) \in \Omega$ *and all* $d \in \mathbb{R}^m$, *then* (3.1) *is FXISS and* $Q(V,x)$ *is an implicit fixed-time ISS Lyapunov function for* (3.1).

Example 3.4

Consider the system

$$\dot{x} = -x^3 + x^2 d_1 - x d_2 + d_1 d_2 \tag{3.7}$$

and the following implicit ISS Lyapunov function candidate:

$$Q(V,x) = \frac{x^2}{2V} - 1. \tag{3.8}$$

We have that $\frac{\partial Q(V,x)}{\partial V} = -\frac{x^2}{2V^2}$, and that $\frac{\partial Q(V,x)}{\partial x} = \frac{x}{V}$, hence

$$-\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) = -\frac{Vx}{2x^2}(-x^3 + x^2 d_1 - x d_2 + d_1 d_2),$$

if $3|d_1| \leq |x|$ and $3|d_2| \leq x^2$ and since $Q = 0 \Rightarrow 1 = \frac{x^2}{2V}$ we have

$$\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \frac{2}{9} x^4.$$

Then, according to Theorem 3.3 the function $Q(V,x)$ defined by (3.8) is an implicit ISS Lyapunov function for (3.7) with $\gamma(|x|) = \frac{2}{9}x^4$ and $\chi(|d|) = \nu^{-1}$, $\nu(r) = \min\{\frac{r}{3}, \frac{r^2}{3}\}$ and we conclude that the origin of (3.7) is ISS. Note that although in this example it is possible to obtain an explicit expression for V, using the implicit framework this is not necessary.

The next example revisits the Example 1.11, where FTS of an unperturbed homogeneous system was determined. Now, instead of using homogeneity properties, we will first present an implicit Lyapunov function candidate from which FTS can be derived. Then we will show that the system is furthermore FTISS. The ILF candidate will be discussed in more detail in Chapter 4, for the moment, only some of its main properties will be used.

Example 3.5

Consider the double integrator system

$$\dot{x} = A_0 x + bu(x) + d, \quad A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad d, x \in \mathbb{R}^2, \tag{3.9}$$

and the following control law

$$u(x) = -k_1 \lceil x_1 \rfloor^{\frac{\eta}{2-\eta}} - k_2 \lceil x_2 \rfloor^{\eta}, \tag{3.10}$$

where $\eta \in (0,1)$ and k_1, k_2 are such that $s^2 + k_2 s + k_1$ is a Hurwitz polynomial. It is clear that for $d = 0$, (3.9) is equivalent to system (1.33). Let us propose the following implicit Lyapunov function candidate:

$$Q(V,x) = x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \tag{3.11}$$

where $P > 0$ and $D_r(V^{-1}) = \begin{pmatrix} V^{-r_1} & 0 \\ 0 & V^{-r_2} \end{pmatrix}$, $r_1 = \frac{2-\eta}{2}, r_2 = \frac{1}{2}$. Let us first show that Q indeed constitutes a well defined ILF candidate. The function $Q(V,x)$ is differentiable for any $(V,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$ and since $P > 0$ then

$$\frac{\lambda_{\min}(P)\|x\|^2}{V} \le Q(V,x) + 1 \le \frac{\lambda_{\max}(P)\|x\|^2}{V^{2-\eta}} \tag{3.12}$$

and there exist some $V^-, V^+ \in \mathbb{R}_+$ such that $Q(V^-,x) < 0 < Q(V^+,x)$ and some $V \in \mathbb{R}_+$ such that $Q(V,x) = 0$. Hence conditions **C1**-**C3** of Theorem 1.13 are fulfilled. Remark that for $\eta = 1$, the identity $Q(V,x) = 0$ defines the quadratic Lyapunov function $V(x) = x^T P x$. The derivative of Q w.r.t. V is given by

$$\partial_V Q = -V^{-1} x^T D_r(V^{-1})(H_r P + P H_r) D_r(V^{-1}) x,$$

where $H_r = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. Since $H_r = \frac{1}{2} I_2 + \frac{1-\eta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ where I_2 is the identity matrix, $H_r \to \frac{1}{2} I_2$ as $\eta \to 1$ and

$$0 < P H_r + H_r P,$$

so that $\partial_V Q(V,x) < 0$ for all $(V,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$ and condition **C4** of Theorem 1.13 is satisfied. Assuming additionally that $P H_r + H_r P \le P$ and taking into account (from (3.11)) that $Q(V,x) = 0 \Rightarrow x^T D_r(\frac{1}{V}) P D_r(\frac{1}{V}) x = 1$ we obtain

$$-V^{-1} \le \partial_V Q(V,x) < 0. \tag{3.13}$$

Similarly, the derivative of Q along the trajectories of (3.9)-(3.10), denoted as $\partial_x Q f$, is given by

$$\partial_x Q f = 2x^T D_r(\tfrac{1}{V}) P D_r(\tfrac{1}{V})(A_0 x + bu(x) + d).$$

Let us assume that the following condition holds for some $\mu > 0$:

$$A_0 S + S A_0^T + bq + b^T q^T + S + \mu I_2 \leq 0, \tag{3.14}$$

where $S = P^{-1}$ and $q = kS^{-1}$. By adding and subtracting the term $2V^{\eta/2} x^T D_r(\frac{1}{V}) P D_r(\frac{1}{V}) bk D_r(\frac{1}{V}) x$, and taking into account that $D_r(\frac{1}{V}) A_0 D_r^{-1}(\frac{1}{V}) = V^{(\eta-1)/2} A_0$ and that $D_r(\frac{1}{V}) b = V^{-\frac{1}{2}} b$, we obtain

$$\partial_x Q f = \begin{pmatrix} y \\ z \end{pmatrix}^T \Theta \begin{pmatrix} y \\ z \end{pmatrix} + V^{\frac{\eta-1}{2}} (2y^T P bk \tilde{y}_\eta - y^T P y) + \tfrac{1}{\mu} V^{\frac{1-\eta}{2}} z^T z,$$

where $y = D_r(\frac{1}{V}) x$, $z = D_r(\frac{1}{V}) d$, $\tilde{y}_\eta = y - (\lceil y_1 \rfloor^{\frac{\eta}{2-\eta}}, \lceil y_2 \rfloor^{\eta})^T$ and

$$\Theta = \begin{pmatrix} V^{\frac{\eta-1}{2}} (P(A_0 - bk) + (A_0 - bk)^T P + P) & P \\ P & -\frac{1}{\mu} V^{\frac{1-\eta}{2}} I_2 \end{pmatrix}.$$

Since the Schur complement of $\begin{pmatrix} P^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \Theta \begin{pmatrix} P^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix}$ for any $\varepsilon \in \mathbb{R}$ is equivalent to the left hand side of (3.14) and $Q(V, x) = 0 \Rightarrow y^T P y = 1$, we have that

$$\partial_x Q f \leq V^{\frac{\eta-1}{2}} (2y^T P bk \tilde{y}_\eta - 1) + \tfrac{1}{\mu} V^{\frac{1-\eta}{2}} d^T D_r^2(\tfrac{1}{V}) d.$$

Since $\tilde{y}_\eta \to 0$ and $D_r^2(\frac{1}{V}) \to V^{-1} I_2$ as $\eta \to 1$, there exists some η, sufficiently close to one, such that $\max_{y: y^T P y = 1} y^T P bk \tilde{y}_\eta < l_1 \leq 1$. Then

$$\partial_x Q < -l_2 V^{\frac{\eta-1}{2}} + \tfrac{1}{\mu} V^{-1 + \frac{1-\eta}{2}} d^T d,$$

where $l_2 = 1 - l_1 > 0, l_1 > \frac{1}{\mu}$. From (3.13) we obtain

$$\frac{\partial_x Q}{\partial_V Q} f(x, d) \geq l_1 V^{1 + \frac{\eta-1}{2}} - \tfrac{1}{\mu} V^{-1 + \frac{1-\eta}{2}} d^T d,$$

and from (3.12) we finally derive

$$\|x\| \geq \chi(\|d\|) \Rightarrow \frac{\partial_x Q}{\partial_V Q} f(x, d) \geq (l_1 - \tfrac{1}{\mu}) V^{1 + \frac{\eta-1}{2}}$$

where $\chi(r) = \frac{1}{\lambda_{\min}(P)} r^{\frac{1}{\eta+1}}$, $1 + \frac{\eta-1}{2} < 1$ and we recover the condition **C5**$^{\text{ft}}$ of Definition 3.5. Thus, we conclude that $Q(V, x)$ is a finite-time implicit ISS Lyapunov function and from Theorem 3.3, the system (3.9)-(3.10) is FTISS for any η sufficiently close to 1. Figure 3.4, shows the simulation plot of system (3.9) with with the disturbance $d_1(t) = d_2(t) = \hat{d}(t)$, where

$$\hat{d}(t) = 0.2\sin(10t) + \begin{cases} 1 & \text{if } t \in [5,6] \\ 0 & \text{otherwise} \end{cases}. \tag{3.15}$$

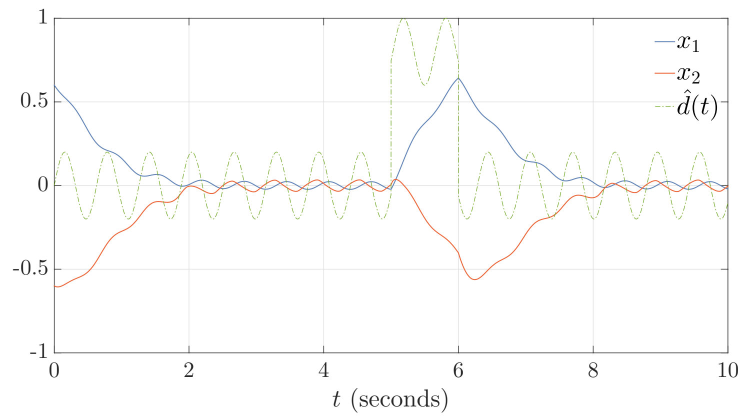

Figure 3.4 – Simulation of system (3.9) for $\eta = 0.8$, initial conditions $x(0) = (0.6, -0.6)$ and the disturbance term (3.15).

3.4 Conclusions

A theoretical framework to characterize NonA ISS has been presented. Both the implicit and the explicit approach have been addressed. For the FTISS property, necessary and sufficient conditions were obtained and although for FXISS only a sufficient condition was presented, it was shown that by introducing some assumptions on the uniformity of the settling-time a converse result can be obtained. The theorems here presented allow to assert, with a single function, the convergence type and the robustness, in an input-to-state sense, of a given nonlinear system. Whenever the disturbances are absent, the results obtained allow to estimate the reaching time to zero. Finally, an alternative dissipativity-like characterization was also provided.

3.5 Proofs

Proof of Theorem 3.1.
Sufficiency. If there exists a finite-time ISS Lyapunov function for (3.1), then we have that $\|x\| \geq \chi(\|d\|)$ implies that

$$DV(x)f(x,d) \leq -cV(x)^{\alpha} \tag{3.16}$$

and from Definition 1.4 we know that (1.20) holds.
I Let us define the set $\mathcal{V} = \{x : V(x) \geq \alpha_2 \circ \chi(\|d\|_{[0,\infty)})\}$. We have that for any $x \in \mathcal{V}$, $\alpha_2(\|x\|) \geq V(x) \geq \alpha_2 \circ \chi(\|d\|_{\infty})$, which implies that $\|x\| \geq \chi(\|d\|)$ and by (3.16), $\mathbb{R}^n \setminus \mathcal{V}$ is an invariant and attractive

set. Then, using the comparison lemma and direct integration, it is straightforward to obtain a class-$\mathcal{GKL}$ function $\beta(r,t)$ such that

$$\|\psi_{x_0}(t,d)\| \leq \beta(\|x_0\|,t) \text{ while } \psi_{x_0}(t,d) \in \mathcal{V}, \tag{3.17}$$

where $\beta(r,t) = 0\ \forall t \geq \widetilde{T}(r)$ and $\widetilde{T}(r)$ is a continuous function for all $r \in \mathbb{R}^n$.

II If $x \notin \mathcal{V}$ then $V(x) < \alpha_2 \circ \chi(\|d\|_{[0,\infty)})$ and therefore $\|x\| \leq \vartheta(\|d\|_{[0,\infty)})$, where $\vartheta = \alpha_1^{-1} \circ \alpha_2 \circ \chi$. In addition, $\mathbb{R}^n \backslash \mathcal{V}$ is invariant so that

$$\|\psi_{x_0}(t,d)\| \leq \vartheta(\|d\|_{[0,\infty)}) \text{ while } \psi_{x_0}(t,d) \notin \mathcal{V}, \tag{3.18}$$

IV Combining (3.17) and (3.18) gives

$$\|\psi_{x_0}(t,d)\| \leq \beta(\|x_0\|,t) + \vartheta(\|d\|_{[0,\infty)})\ \forall t \geq 0,$$

and FTISS for (3.1) is obtained.

Necessity. This part of the proof has four main steps. First, using converse arguments, a Lyapunov function $V(x)$ is constructed that shows FTS of the unperturbed system (3.1). Second, it is shown that this Lyapunov function $V(x)$ is actually an FTISS Lyapunov function if $\|x\| < \rho$, for any $\rho > 0$ (with the asymptotic gain dependent on ρ). Third, applying smoothing tools another ISS Lyapunov function $W(x)$ is designed for $\|x\| > \delta$ for any $\delta \in (0,\rho)$. Finally, a desired global FTISS Lyapunov function is constructed by uniting V and W.

I Since (3.1) is FTISS, when $d = 0$ there exists some $T(x)$ such that $\|\psi_x(t,0)\| = 0\ \forall t \geq T(x)$. If $T(x)$ is a locally Lipschitz function, then by Theorem 1.6, it is possible to define a function $V(x) := T(x)^{\frac{1}{1-\alpha}}$, with $\alpha \in [0,1)$, satisfying $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and such that

$$\frac{\partial V(x)}{\partial x} f(x,0) \leq -cV(x)^\alpha \tag{3.19}$$

for some $c > 0$ and for almost all $x \in \mathbb{R}^n$.

II (Case $\|x\| < \rho$). Since $T(x)$ is Lipschitz continuous, $V(x)$ is also locally Lipschitz continuous and $\|\frac{\partial V}{\partial x}\| \leq \kappa + \eta(\|x\|)$ for some $\kappa \in \mathbb{R}_{\geq 0}$ and $\eta \in \mathcal{K}$. By Assumption 1, $\|f(x,d) - f(x,0)\| \leq L(\|x\|)\|d\|$ for some $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Thus

$$\left\| \frac{\partial V(x)}{\partial x} \Big(f(x,d) - f(x,0) \Big) \right\| \leq (\kappa + \eta(\|x\|)) L(\|x\|)\|d\|.$$

Let us define

$$d := \varphi_\rho(\|x\|)u \text{ where } \varphi_\rho(\|x\|) := \frac{c\,\alpha_1(\|x\|)^\alpha}{2(\kappa + \eta(\rho))L(\rho)}. \tag{3.20}$$

Then, for $\|u\| \leq 1$ and some $\rho > 0$, it becomes clear that $\|x\| \leq \rho$ implies

$$\left\| \frac{\partial V(x)}{\partial x} \Big(f(x, \varphi_\rho(\|x\|)u) - f(x,0) \Big) \right\| \leq \tfrac{c}{2} V(x)^\alpha.$$

From (3.20), it follows that $\|u\| \leq 1$ implies that $\|x\| \geq \varphi_\rho^{-1}(\|d\|)$ and using the inequality (3.19) we have that $\rho > \|x\| \geq \varphi_\rho^{-1}(\|d\|)$ implies

$$\frac{\partial V(x)}{\partial x} f(x,d) \leq -\tfrac{c}{2} V(x)^\alpha.$$

III (Case $\|x\| > \delta$). For any two constants $L_x > 0$ and $L_u > 0$ define the function

$$\mu_\delta(a,d) := \min\left\{1, \frac{(L_x a + L_d\|d\|)(1+\sup_{\|x\|\leq a}\|f(x,d)\|)}{(L_x\delta + L_d\|d\|)(1+\sup_{\|x\|\leq \delta}\|f(x,d)\|)}\right\},$$

for some $\delta \in (0,\rho)$. Note that by design μ_δ is continuous, increasing, bounded by 1, equals to 1 when $a = \delta$, strictly positive outside of the origin and $\mu_\delta(0,0) = 0$. Define a vector field

$$f_\delta(x,d) := \begin{cases} f(x,d), & \text{if } \|x\| \geq \delta \\ \mu_\delta(\|x\|,d) f(x,d), & \text{if } \|x\| < \delta \end{cases},$$

which is locally Lipschitz and continuous by construction. Indeed, the function f possesses this property outside of the origin by the imposed hypothesis, and for $\|x\| < \delta$ we have that

$$\begin{aligned}
\|f_\delta(x,d)\| &\leq \|\mu_\delta(\|x\|,d) f(x,d)\| \\
&= \frac{\|f(x,d)\|}{1+\sup_{\|s\|\leq\delta}\|f(s,d)\|} \cdot \frac{1+\sup_{\|s\|\leq\|x\|}\|f(s,d)\|}{L_x\delta + L_d\|d\|}(L_x\|x\| + L_d\|d\|) \\
&\leq \frac{1+\sup_{\|s\|\leq\|x\|}\|f(s,d)\|}{L_x\delta + L_d\|d\|}(L_x\|x\| + L_d\|d\|) \\
&\leq \frac{1+\sup_{\|s\|\leq\delta}\|f(s,d)\|}{L_x\delta + L_d\|d\|}(L_x\|x\| + L_d\|d\|),
\end{aligned}$$

so that f_δ is locally Lipschitz for all $x \in \mathbb{R}^n$. Now let us consider the system

$$\dot{x} = f_\delta(x,d),$$

where f_δ is, as showed above, a locally Lipschitz continuous function and it is ISS since (3.1) has this property (multiplication by a continuous strictly positive function μ_δ does not influence the stability, it acts as a time re-scaling). Consider now the following modified version of the system (3.1):

$$\dot{x} = f_\delta(x,d) = f(x,d) + \Delta f,$$

where $\Delta f := f(x,d) - f_\delta(x,d)$, and by construction $\|x\| \geq \delta \Rightarrow \Delta f = 0$. Following the converse results on existence of ISS Lyapunov functions, there exists a continuously differentiable, positive definite and radially unbounded function $W:\mathbb{R}^n_{\geq 0}\to\mathbb{R}_{\geq 0}$, $\alpha_3\in\mathcal{K}_\infty$ and $\sigma\in\mathcal{K}$ such that

$$\|x\|\geq\sigma(\|d\|)\Rightarrow\tfrac{\partial W(x)}{\partial x}f_\delta(x,d)\leq-\alpha_3(\|x\|),$$

then due to the properties of the auxiliary perturbation Δf:

$$\|x\|\geq\max\{\delta,\sigma(\|d\|)\}\Rightarrow\tfrac{\partial W(x)}{\partial x}f(x,d)\leq-\alpha_3(\|x\|).$$

IV Let us define the function

$$\widetilde{V}(x):=s(V(x))W(x)+(1-s(V(x)))V(x),$$

where $s:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ satisfies

$$s(r)=\begin{cases}1 & \text{if } r\geq\alpha_2(\rho)\\ 0 & \text{if } r\leq\alpha_1(\delta)\end{cases}$$

and $s'(r)=\frac{\partial s(r)}{\partial r}>0$ for all $r\in(\alpha_1(\delta),\alpha_2(\rho))$. Assume that $V(x)\leq W(x)$ for all $x\in\{x\in\mathbb{R}^n:\alpha_1(\delta)\leq V(x)\leq\alpha_2(\rho)\}$ (both functions, $V(x)$ and $W(x)$, are continuous, positive definite and radially unbounded, then we can adopt such a hypothesis without being restrictive, since multiplying $W(x)$ by a constant we can always assure its fulfillment), then we have that

$$\tfrac{\partial\widetilde{V}(x)}{\partial x}f(x,d)=s\dot{W}(x)+(1-s)\dot{V}(x)+s'\dot{V}(W(x)-V(x)),$$

and gathering all the previous estimates, we arrive to

$$\|x\|\geq\chi(\|d\|)\Rightarrow\tfrac{\partial\widetilde{V}(x)}{\partial x}f(x,d)\leq-\alpha_4(\|x\|), \tag{3.21}$$

where $\chi(r):=\max\{\sigma(r),\varphi_\rho^{-1}(r)\}$ and $\alpha_4\in\mathcal{K}_\infty$ such that

$$\alpha_4(\|x\|)\geq\begin{cases}\alpha_3(\|x\|), & V(x)\geq\alpha_2(\rho)\\ \frac{c}{2}\,\widetilde{V}(x)^\alpha, & V(x)\leq\alpha_1(\delta)\end{cases}.$$

Consequently, $\widetilde{V}$ is a finite-time ISS Lyapunov function for (3.1). ∎

Proof of Theorem 3.2. The proof follows closely the reasoning of the sufficiency proof of Theorem 3.1. Instead of (3.19), the estimate

$$DV(x)f(x,d)\leq-c_1V(x)^\alpha-c_2V(x)^\beta$$

is obtained. Then, form Theorem 1.3 we know that the inequality (3.5) holds with $\beta(r,t)=0\ \forall t\geq\widetilde{T}(r)$ and that $\sup_{r\in\mathbb{R}_{\geq 0}}\widetilde{T}(r)<+\infty$. Since the estimates (3.17) and (3.18) also hold in this case, we conclude FXISS of the origin of (3.1). ∎

Proof of Theorem 3.3.

I Conditions **C1**, **C2** and **C4** of Theorem 1.13, and the implicit function theorem imply that the equation $Q(V,x) = 0$ implicitly defines a unique function $V : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}_+$ such that $Q(V(x),x) = 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$.

II The function V is continuously differentiable outside the origin and $\partial_x V = -\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)}$ for $Q(V,x) = 0$, $x \neq 0$. Condition **C3** of Theorem 1.13 implies that the function V can be continuously prolonged at the origin (by setting $V(0) = 0$) and that V is positive definite and radially unbounded; by Lemma 1.2 this means that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{3.22}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and therefore (1.20) holds.

III The derivative of V along the vector field (3.1) is given by

$$\begin{aligned} DV(x)f(x,d) &= \partial_x V f(x,d) \\ &= -\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d); \end{aligned}$$

and from condition **C5**$^{\text{iss}}$ we obtain

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -\gamma(\|x\|),$$

for all $(V,x) \in \Omega$. Therefore $Q(V,x) = 0$ implicitly defines an ISS Lyapunov function for system (3.1). Consequently, according to Theorem 1.13, (3.1) is an ISS system.

Converse implication. If system (3.1) is an ISS system, then there exists an ISS Lyapunov function $\tilde{V}: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for it (see Theorem 1.7). Then it is clear that the implicit ISS Lyapunov function $Q(V,x) = \frac{\tilde{V}(x)}{V} - 1$ satisfying **C1**-**C5**$^{\text{iss}}$ also exists. ■

Proof of Corollary 3.1. The proof is a direct consequence of Theorem 3.3 and Lemma 1.3. ■

Proof of Theorem 3.4.

Sufficiency. **I** As shown before, from conditions **C1**-**C4** of Theorem 1.13, $Q(V(x),x) = 0$ implicitly defines a unique, proper, positive definite function $V(x)$ such that (3.22) holds and its derivative along (3.1) is given by

$$DV(x)f(x,d) = -\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d). \tag{3.23}$$

II From condition **C5**$^{\text{ft}}$ and (3.23) we have that $\|x\| \geq \chi(\|d\|)$ implies that

$$DV(x)f(x,d) \leq -cV^a(x)$$

so that Q implicitly defines a finite-time ISS Lyapunov function. The result follows by applying Theorem 3.1.

Necessity. From Theorem 3.1, if (3.1) is FTISS, there exist a finite-time ISS Lyapunov function $\tilde{V}$.

Then it is straightforward to construct an implicit finite-time ISS Lyapunov Q, *e.g.* $Q = \frac{\tilde{V}}{V} - 1$, that satisfies conditions **C1**-$\mathbf{C5}^{\mathrm{ft}}$ hold. ∎

Proof of Theorem 3.5. By previous considerations and if condition $\mathbf{C5}^{\mathrm{fx}}$ holds, then for $\|x\| \geq \chi(\|d\|)$

$$DV(x)f(x,d) \leq -c_1 V(x)^{\alpha} - c_2 V(x)^{\beta},$$

so that Q implicitly defines a fixed-time ISS Lyapunov function and from Theorem 3.2 we conclude that the origin of (3.1) is FXISS. ∎

Proof of Corollary 3.2. From $\mathbf{C5}^{\mathrm{ft}^*}$ and previous considerations we have that

$$DV(x)f(x,d) \leq \delta(\|d\|) - \kappa V(x)^{\alpha}.$$

by adding and subtracting $\theta V(x)^{\alpha}$, with $\theta \in (0,\kappa)$ we obtain

$$DV(x)f(x,d) \leq -(\kappa - \theta)V(x)^{\alpha} - \theta V(x)^{\alpha} + \delta(\|d\|),$$

and by taking into account (3.22), it becomes clear that it is always possible to find some $\chi \in \mathcal{K}_{\infty}$ such that

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -cV(x)^{\alpha},$$

take, for instance, $c = \kappa - \theta$ and $\chi = \alpha_2^{-1} \circ (\frac{1}{\theta}\delta)^{1/\alpha}$. The fixed-time case can be dealt with by following the same reasoning and repeating the arguments of the proof of Theorem 3.4. ∎

Chapter 4

FT and FxT Observers for Linear MIMO Systems via ILF

Contents

Let us consider the perturbed linear control system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + d_x(t), \\ y(t) = Cx(t) + d_y(t), \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state variable, $y \in \mathbb{R}^k$ is the measured output, $u : \mathbb{R} \to \mathbb{R}^s$ is the control input, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times s}$ is the matrix of input gains and the matrix $C \in \mathbb{R}^{k\times n}$ is the output matrix which links the measured outputs to the state variables. The pair (A, C) is assumed to be observable and $\text{rank}(C) = k$.

The goal of this chapter is twofold:

◇ Design two dynamic observers that estimate the state of the non-perturbed system (4.1) in a finite time or in a fixed (defined *a priori*) time, under the assumption that the domain of initial conditions is unknown.

◇ Both observers must be robust (in an *input-to-state* sense) with respect to $\mathcal{L}_\infty$-bounded measurement noises $d_y(t)$ and $\mathcal{L}_\infty$-bounded disturbances $d_x(t)$.

The theorems to be presented will be used for analysis and design of finite-time and fixed-time observers using the implicit Lyapunov approach. In this chapter and the one that follows we will make use of a particular type of implicit LF candidate, whose main properties will be discussed in detail.

4.1 Implicit Lyapunov Function Candidate

The corresponding implicit LF candidate is selected as

$$Q(V,z) := z^T D_r(V^{-1}) P D_r(V^{-1}) z - 1, \tag{4.2}$$

where $V \in \mathbb{R}_+$, $z \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $D_r(\cdot)$ is the dilation matrix of the form

$$D_r(\lambda) = \mathrm{diag}(\lambda^{r_1} I_{n_1}, \lambda^{r_2} I_{n_2}, \dots, \lambda^{r_m} I_{n_m}), \tag{4.3}$$

with $r = (r_1, ..., r_m)^T \in \mathbb{R}^m, r_i > 0$, $r_{\min} = \min_{1\le j\le n} r_j$ and n_i are natural numbers such that $n_1 + ... + n_m = n$.

Several remarks about this particular selection of candidate function are appropiate.

- ⋄ The function Q is an implicit analog of the quadratic Lyapunov function. Indeed, for $r_1 = ... = r_m = 0.5$, $Q(V,z) = 0$ implies that $V = z^T P z$.
- ⋄ If $m = 2$, $n_1 = n_2 = 1$ and $r_1 = 2, r_2 = 1$, V can be found analically. Indeed, the equation $Q(V,x) = 0$ becomes
$$V^4 - p_{22} x_2^2 V^2 - 2p_{12} x_1 x_2 V - p_{11} x_1^2 = 0,$$
where $\{p_{ij}\}$ are elements of the matrix $P > 0$ and $(x_1, x_2) \in \mathbb{R}^2$. The roots of this equation can be found using, for example, Ferrari formulas. For higher dimensional systems and/or for a different selection of r, it becomes much more difficult to obtain V analically. Nonetheless, using numerical methods, such as the bisection method, V can be easily found numerically [Polyakov et al., 2015].
- ⋄ In [Polyakov et al., 2015] the implicit framework is used to design FT and FxT controllers for a chain of integrators. The structure of the controllers involves online calculation of V, which makes its implemantation computationally expensive[1]. In the observation algorithms to be presented, online calculation of V is not needed and the parameters and the gain matrices are calculated offline.
- ⋄ As will be seen, the particular structure of Q allows to tranform the conditions for FTS and FxTS of Theorems 1.14 and 1.15 into an LMI feasability problem. Moreover, the selection of r will influence the settling-time estimates, allowing to adjust to some extent the convergence rate.
- ⋄ With a suitable selection of r, $Q(V,x) = 0$ implicitly defines a homogeneous function V (see [Polyakov et al., 2014] for more details).

[1] An improvement of this work, where the online calculation of V is no longer needed can be found in [Zimenko et al., 2018].

Block Diagonal Decomposition

The observers' design starts with a decomposition of the considered system into an appropriate block canonical form.

Lemma 4.1. *Consider the system* (4.1) *with the pair* (A, C) *being observable and* $\text{rank}(C) = k$*. Then there exists a nonsingular transformation* Φ *such that*

$$\Phi A \Phi^{-1} = F\tilde{C} + \tilde{A}, \quad C\Phi^{-1} = \left(C_0 \; 0 \; \dots \; 0 \right), \quad \tilde{C} = (I_k \; 0) \in \mathbb{R}^{k \times n},$$

$$\tilde{A} = \begin{pmatrix} 0 & A_{1\,2} & 0 & \dots & 0 \\ 0 & 0 & A_{2\,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{m-1\,m} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \; C_0 \in \mathbb{R}^{k \times k}, F \in \mathbb{R}^{n \times k},$$

where m *is an integer,* $A_{j-1\,j} \in \mathbb{R}^{n_{j-1} \times n_j}$*,* $n_j = \text{rank}(A_{j-1\,j})$*,* $j = 2, \dots, m$*, so that* $n_1 = \text{rank}(C) = k$ *and* $\sum_{i=1}^{m} n_i = n$*.*

We omit the proof of this lemma since it is a consequence of well known results on block observability and controllability forms (see [Wonham, 1974], [Drakunov et al., 1990a], [Drakunov et al., 1990b], [Misrikhanov and Ryabchenko, 2011]), however, we provide a suitable algorithm to calculate the transformation matrix Φ in the Proofs section. If $k = 1$ then $m = n$, $n_i = 1$ and Φ transforms the matrix A into the canonical Brunovsky form. It is also worth stressing that canonical forms and related transformations also exist for nonlinear systems (see for instance [Isidori, 2013], [Khalil and Praly, 2014]). Therefore, the observer design algorithms given below can be adapted to the nonlinear case.

4.2 Observers' Design

Finite-Time Observer

Let us consider the following nonlinear observer

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) - g_{_{FT}}(y(t) - C\hat{x}(t)), \tag{4.4}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector and the function $g_{_{FT}} : \mathbb{R}^k \to \mathbb{R}^n$ is defined as

$$g_{_{FT}}(\sigma) := \Phi^{-1}\left[D_{\tilde{r}}\left(\|\tilde{P}C_0^{-1}\sigma\|^{-1}\right)L_{_{FT}} - F\right]C_0^{-1}\sigma, \tag{4.5}$$

where $\sigma \in \mathbb{R}^k$, the matrices $\Phi \in \mathbb{R}^{n \times n}$, $C_0 \in \mathbb{R}^{k \times k}$ and $F \in \mathbb{R}^{n \times k}$ are defined in Lemma 4.1, $D_{\tilde{r}}(\cdot)$ is the dilation matrix given by (4.3) with

$$\tilde{r} = \left(\tfrac{\mu}{1+(m-1)\mu}, \; \tfrac{2\mu}{1+(m-1)\mu}, \dots, \tfrac{m\mu}{1+(m-1)\mu}\right)^T, \; \mu \in (0, 1], \tag{4.6}$$

and $L_{FT} \in \mathbb{R}^{n\times k}$ and $\tilde{P} \in \mathbb{R}^{k\times k}$ are matrices of observer gains, to be determined. The error equation in the *disturbance-free case* (*i.e.* $d_x = 0$, $d_y = 0$) has the form

$$\dot{e} = \left(\tilde{A} + D_{\tilde{r}}\left(\|\tilde{P}\tilde{C}e\|^{-1}\right)L_{FT}\tilde{C}\right)e, \tag{4.7}$$

where $e = \Phi(x-\hat{x})$, $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{C} \in \mathbb{R}^{k\times n}$ are defined in Lemma 4.1. Obviously, if $\mu \to 0$ then $D_{\tilde{r}}\left(\|\tilde{P}\tilde{C}e\|^{-1}\right) \to I_n$ and the presented observer becomes the classical Luenberger one.

Remark 4.1. If the term $\|\tilde{P}C_0^{-1}\sigma\|$ in (4.5) is replaced by $\epsilon^{(1+(m-1)\mu)/\mu}$ where $\epsilon > 0$ is a small constant, then the system (4.4)-(4.5) becomes a high-gain observer [Prasov and Khalil, 2013], [Khalil and Praly, 2014] with the error dynamics given by

$$\dot{e} = \left(\tilde{A} + \mathrm{diag}(\epsilon^{-1}I_{n_1}, \epsilon^{-2}I_{n_2}, \cdots, \epsilon^{-m}I_{n_m})L_{FT}\tilde{C}\right)e.$$

In our algorithms the gain factor ϵ depends on the available part of the observation error, namely, on $\sigma = y - C\hat{x} = Ce$. This allows the finite-time and fixed-time observers to be less sensitive with respect to noises (see Section 4.3).

Let us define $H_r = \mathrm{diag}(r_1 I_{n_1}, r_2 I_{n_2}, ..., r_m I_{n_m}) \in \mathbb{R}^{n\times n}$, $r = \left[1 + \frac{\mu}{1+(m-1)\mu}\right]\mathbb{1}_m - \tilde{r}$ and $\Xi(\lambda) = \lambda\left(D_{\tilde{r}}(\lambda^{-1}) - I_n\right)$.

Theorem 4.1. *Let for some* $\mu \in (0,1]$, $\alpha > 0$, $\xi > 0$ *and* $\tau \geq 1$ *the system of matrix inequalities*

$$\begin{pmatrix} P\tilde{A}+\tilde{A}^TP+\tilde{C}^TY^T+Y\tilde{C}+\xi P+\alpha(PH_r+H_rP) & P \\ P & -\xi Z \end{pmatrix} \leq 0, \tag{4.8a}$$

$$P > 0,\ Z > 0,\ X > 0, \tag{4.8b}$$

$$\begin{pmatrix} \tau X & Y^T \\ Y & P \end{pmatrix} \geq 0, \tag{4.8c}$$

$$P \geq \tilde{C}^T X \tilde{C}, \tag{4.8d}$$

$$PH_r + H_r P > 0, \tag{4.8e}$$

$$\Xi(\lambda) Z \Xi(\lambda) \leq \tfrac{1}{\tau} P, \quad \forall \lambda \in [0,1], \tag{4.8f}$$

be feasible for some $P, Z \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{k\times n}$ *and* $X \in \mathbb{R}^{k\times k}$. *Then the error equation* (4.7) *with* $L_{FT} = P^{-1}Y$ *and* $\tilde{P} = X^{1/2}$ *is globally finite-time stable with settling time* $T \leq \frac{V^\rho(e(0))}{\alpha\rho}$, $\rho = \frac{\mu}{1+(m-1)\mu}$, *where* $V : \mathbb{R}^n \to \mathbb{R}$ *is defined implicitly by the equation* $Q(V,\cdot) = 0$ *with* Q *given by* (4.2).

In other words, this theorem claims that any solution of the observer system (4.4) converges to a solution of the real system (4.1) in a finite time T, which is dependent on the initial estimation error $e(0) \in \mathbb{R}^n$. The main idea of the proof is to show that the function Q (defined in the statement of Theorem 4.1) satisfies all conditions of Theorem 1.14. Proofs of all theorems and propositions are given in the Proofs section.

Corollary 4.1. *The system of matrix inequalities* (4.8) *is feasible for sufficiently small* $\mu > 0$.

Indeed, observability of the pair (A,C) implies that the pair $(\tilde{A}+0.5(\alpha H_r+\xi I_n),\tilde{C})$ is also observable. Hence, it can be easily shown that the inequality (4.8a) is feasible with some positive definite matrices $P,Z\in\mathbb{R}^{n\times n}$ and $Y\in\mathbb{R}^{k\times n}$. The matrix inequalities (4.8d) and (4.8c) are also feasible for some $X\in\mathbb{R}^{k\times k}$ and sufficiently large $\tau\geq 1$. Since $\|\Xi(\lambda)\|\to 0$ uniformly on $\lambda\in[0,1]$ as $\mu\to 0$, then the inequalities (4.8e), (4.8f) will hold for sufficiently small $\mu>0$.

In order to apply Theorem 4.1 we need to solve the parametrized system of nonlinear matrix inequalities (4.8) with respect to variables P, X, Z, Y for a given $\mu\in(0,1]$ and $\alpha,\xi,\tau>0$. By fixing the value $\lambda\in[0,1]$, the system (4.8) becomes a system of LMIs, which can be solved using any appropriate mathematical software (*e.g.* MATLAB). However, the mentioned LMIs must be checked for any $\lambda\in[0,1]$. Due to the smoothness of $\Xi(\lambda)$ with respect to $\lambda\in(0,1]$, this can be done on a proper grid constructed over this interval. The next corollary provides sufficient feasibility conditions for the parametrized matrix inequality (4.8f).

Proposition 4.1. *The parametric inequality* (4.8f) *holds if*

$$\Xi(q_i)Z\Xi(q_i)+\tfrac{1}{2}(q_i^2-q_{i-1}^2)M<\tfrac{1}{\tau}P,\ i=1,\dots,N, \tag{4.9a}$$

$$P>0,\quad Z>0,\quad M>0 \tag{4.9b}$$

$$\begin{pmatrix}(I_n-H_{\tilde{r}})Z+Z(I_n-H_{\tilde{r}}) & -ZH_{\tilde{r}}\\ -H_{\tilde{r}}Z & M\end{pmatrix}>0, \tag{4.9c}$$

where $0=q_0<q_1<\ldots<q_N=1$, $H_{\tilde{r}}=\mathrm{diag}(\tilde{r}_1I_{n_1},\tilde{r}_2I_{n_2},\dots,\tilde{r}_mI_{n_m}),P,M,Z\in\mathbb{R}^{n\times n}$.

The provided result allows the implementation of a simple algorithm to solve the parametrized system of matrix inequalities (4.8) with fixed α, ξ, τ and μ.

Algorithm 4.1.

Initialization: $N=1,q_0=0,q_N=1,\Sigma=\{q_0,q_N\}$.

Loop: *While the system of LMIs* (4.8a-4.8d), (4.9) *is not feasible, do* $\Sigma\leftarrow\Sigma\cup\left\{\frac{q_{i-1}+q_i}{2}\right\}_{i=1}^{N}$ *and* $N\leftarrow 2N$.

Since the matrix inequality $(I_n-H_{\tilde{r}})Z+Z(I_n-H_{\tilde{r}})>0$ is obviously feasible for sufficiently small $\mu>0$, then, in the view of Corollary 4.1, the presented algorithm always finds the required solution if μ is sufficiently small.

Fixed-Time Observer

Let us consider now the observer

$$\frac{d}{dt}\hat{x}(t)=A\hat{x}+Bu(t)-g_{FX}(y(t)-C\hat{x}(t)), \tag{4.10}$$

where $\hat{x}\in\mathbb{R}^n$ and the function $g_{FX}:\mathbb{R}^k\to\mathbb{R}^n$ is defined as

$$g_{FX}(\sigma):=\Phi^{-1}\Big[\tfrac{1}{2}\Big\{D_{\tilde{r}}\Big(\|\tilde{P}_1C_0^{-1}\sigma\|^{-1}\Big)+D_{\tilde{r}}\Big(\|\tilde{P}_2C_0^{-1}\sigma\|\Big)\Big\}L_{FX}-F\Big]C_0^{-1}\sigma,$$

where $\sigma\in\mathbb{R}^k$, the matrices $\Phi\in\mathbb{R}^{n\times n}$, $C_0\in\mathbb{R}^{k\times k}$ and $F\in\mathbb{R}^{n\times k}$ are defined in Lemma 4.1, and the matrices $\tilde{P}_i\in\mathbb{R}^{k\times k}$, $i=1,2$ and $L_{FX}\in\mathbb{R}^{n\times k}$ are gain matrices to be determined.

The error equation between (4.1) and (4.10) with $d_x = d_y = 0$ is given by

$$\dot{e} = \left(\tilde{A} + \tfrac{1}{2}\left\{D_{\tilde{r}}\left(\|\tilde{P}_1\tilde{C}e\|^{-1}\right) + D_{\tilde{r}}\left(\|\tilde{P}_2\tilde{C}e\|\right)\right\}L_{FX}\tilde{C}\right)e, \tag{4.11}$$

$e = \Phi(x-\hat{x})$, where $\tilde{A} \in \mathbb{R}^{n\times n}$ and $\tilde{C} \in \mathbb{R}^{k\times n}$ are defined in Lemma 4.1.

Let us define $r_i = (-1)^i\tilde{r} + \left[1 + \frac{(-1)^{i+1}\mu}{1+(m-1)\mu}\right]\mathbb{1}_m$, $\bar{\Xi}_i^\delta(\lambda) = \frac{\lambda_1}{2}\left\{D_{\tilde{r}}\left(\frac{\lambda_2^{i-1}}{\delta_1\lambda_1}\right) + D_{\tilde{r}}\left(\frac{\delta_2\lambda_1}{\lambda_2^{i-2}}\right) - 2I_n\right\}$, $\delta = (\delta_1,\delta_2)$, $\lambda = (\lambda_1,\lambda_2)$ and $H_i = \mathrm{diag}\{(r_i)_1 I_{n_1},\ldots,(r_i)_m I_{n_m}\} \in \mathbb{R}^{n\times n}$ for $i = 1,2$.

Theorem 4.2. *Let for some $\mu \in (0,1]$, $\alpha > 0$, $\xi > 0$, $\tau \geq 1$ and $\delta = (\delta_1,\delta_2), \delta_i > 0, i = 1,2$, the system of matrix inequalities*

$$\begin{pmatrix} P\tilde{A} + \tilde{A}^T P + \tilde{C}^T Y^T + Y\tilde{C} + \xi P + \alpha(PH_i + H_i P) & P \\ P & -\xi Z_i \end{pmatrix} \leq 0, \tag{4.12a}$$

$$\begin{pmatrix} \tau X & Y^T \\ Y & P \end{pmatrix} \geq 0, \tag{4.12b}$$

$$P > 0, X > 0, Z_i > 0, \tag{4.12c}$$

$$PH_i + H_i P > 0, \tag{4.12d}$$

$$P \geq \tilde{C}^T X \tilde{C}, \tag{4.12e}$$

$$\bar{\Xi}_i^\delta(\lambda) Z_i \bar{\Xi}_i^\delta(\lambda) \leq \tfrac{1}{\tau}P, \quad \forall \lambda \in [0,1]\times[0,1], \tag{4.12f}$$

be feasible with $P, Z_1, Z_2 \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n_1\times n}$ and $X \in \mathbb{R}^{k\times k}$. Then the error equation (4.11) with $L_{FX} = P^{-1}Y$, $\tilde{P}_i = \delta_i X^{1/2}$ is globally fixed-time stable with $T_{\max} \leq \frac{1+(m-1)\mu}{0.5\alpha\mu}$.

Under the following additional restrictions, the parametric LMI (4.12f) can be simplified:

Proposition 4.2. *Let $0 = q_0 < q_1 < \ldots < q_{N_1} = 1$ and $0 < p_0 < p_1 < \ldots < p_{N_2} = 1$ for some $N_1, N_2 \geq 1$. If the positive definite matrices $Z_i, S_i, M_i, R_i, U_i \in \mathbb{R}^{n\times n}$ satisfy the following system of LMIs*

$$\begin{pmatrix} Z_i H_{\tilde{r}} + H_{\tilde{r}} Z_i & Z_i H_{\tilde{r}} \\ H_{\tilde{r}} Z_i & U_i \end{pmatrix} \geq 0, \tag{4.13a}$$

$$M_i H_{\tilde{r}} + H_{\tilde{r}} M_i > 0, \tag{4.13b}$$

$$\begin{pmatrix} 2Z_i - Z_i H_{\tilde{r}} - H_{\tilde{r}} Z_i & Z_i H_{\tilde{r}} & -Z_i H_{\tilde{r}} \\ H_{\tilde{r}} Z_i & M_i & 0 \\ -H_{\tilde{r}} Z_i & 0 & S_i \end{pmatrix} \geq 0, \tag{4.13c}$$

$$\bar{\Xi}_i^\delta(q_j,p_s) Z_i \bar{\Xi}_i^\delta(q_j,p_s) + \ln\left(\tfrac{p_s}{p_{s-1}}\right)\bar{\Xi}_i^\delta(q_j,0) U_i \bar{\Xi}_i^\delta(q_j,0) + \tfrac{q_j^2 - q_{j-1}^2}{2}(D_{\tilde{r}}(\delta_2) M_i D_{\tilde{r}}(\delta_2) + S_i) \leq \tfrac{1}{\tau}P, \tag{4.13d}$$

$$\begin{pmatrix} \Upsilon_i(q_j,p_0) & \bar{\Xi}_i^\delta(q_j,0) Z_i D_{\tilde{r}}\left(p_0^{1/2}\right) \\ D_{\tilde{r}}\left(p_0^{1/2}\right) Z_i \bar{\Xi}_i^\delta(q_j,0) & R_i \end{pmatrix} \geq 0, \tag{4.13e}$$

where $\Upsilon_i(q_j,p_0) := \frac{1}{\tau}P - \tilde{D}(q_j)(D_{\tilde{r}}(p_0) Z_i D_{\tilde{r}}(p_0) + D_{\tilde{r}}(p_0^{1/2}) R_i D_{\tilde{r}}(p_0^{1/2}))\tilde{D}(q_j) - \bar{\Xi}_i^\delta(q_j,0) Z_i \bar{\Xi}_i^\delta(q_j,0) - \frac{q_j^2 - q_{j-1}^2}{2}(M_i + S_i)$ and $\tilde{D}_i(q_j) := \frac{q_j}{2} D_{\tilde{r}}\left((\delta_{3-i} q_j)^{(-1)^i}\right)$, $i = 1,2$, $j = 1,\ldots,N_1$, $s = 1,\ldots,N_2$, then (4.12f) holds.

Based on this proposition, an algorithm for solving the parametrized system of LMIs (4.12) can be presented, analogously to the finite-time case.

Algorithm 4.2.
Initialization: $\beta > 0$, $N_1 = 1$, $N_2 = 1$, $p_0 > 0$, $p_{N_1} = 1$, $q_0 = 0$, $q_{N_2} = 1$, $\Sigma = \{q_0, q_{N_1}\}$, $\Lambda = \{p_0, p_{N_2}\}$.
Loop: *While the system of LMIs* (4.12a-4.12e), (4.13) *with* $q_j \in \Sigma$, $p_s \in \Lambda$ *is not feasible, do*
$\Sigma \leftarrow \Sigma \cup \left\{0.5(q_{j-1} + q_j)\right\}_{j=1}^{N_1}$ *with* $q_j \in \Sigma$, $\quad N_1 \leftarrow 2N_1$,
$\Lambda \leftarrow \Lambda \cup \left\{p_1 e^{-\beta}\right\}$ *with* $p_1 \in \Lambda$, $\ N_2 \leftarrow N_2 + 1$.

Remark that the grid Λ is a logarithmic grid such that the term $\ln\left(\frac{p_s}{p_{s-1}}\right)$ in (4.13d) equals β for any $p_s \in \Lambda \backslash \{p_0, p_1\}$.

Robustness Analysis

We consider now (4.1) with nonzero $d_x : \mathbb{R}_+ \to \mathcal{L}_\infty(\mathbb{R}^n)$, and nonzero $d_y : \mathbb{R}_+ \to \mathcal{L}_\infty(\mathbb{R}^k)$. Since the observers' robustness follows from their homogeneity properties, we will establish, for each observer, the type of homogeneity that it exhibits and next that they are robust against bounded disturbances and bounded measurement noise. Again, the error variable is defined as $e = \Phi(x - \hat{x})$.

Corollary 4.2. *Consider the perturbed error equation between* (4.1) *and* (5.5)

$$\dot{e} = \tilde{A}e + D_{\tilde{r}}(\|\tilde{P}\tilde{C}e + d_y\|^{-1})L_{FT}(\tilde{C}e + d_y) + \phi, \tag{4.14}$$

where $\phi = -F\tilde{C}d_y + \Phi d_x$, $d_x \in \mathcal{L}_\infty(\mathbb{R}^n), d_y \in \mathcal{L}^\infty(\mathbb{R}^k)$ *and assume that all conditions of Theorem 4.1 are satisfied. Then the system* (4.14) *is ISS for* $\mu \in (0,1)$ *and iISS for* $\mu = 1$.

A similar result can be provided for the fixed-time observer:

Corollary 4.3. *Consider the following perturbed error equation between* (4.1) *and* (4.10)

$$\dot{e} = \tilde{A}e + \tfrac{1}{2}\{D_{\tilde{r}}(\epsilon_1^{-1}) + D_{\tilde{r}}(\epsilon_2)\}L_{FX}(\tilde{C}e + d_y) + \phi, \tag{4.15}$$

where $\phi = \Phi d_x - F\tilde{C}d_y$, $\epsilon_i = \|\tilde{P}_i\tilde{C}e + d_y\|, i = 1,2$, $d_x, d_y \in \mathcal{L}^\infty$ *and assume that all conditions of Theorem 4.2 are satisfied. Then the error dynamics* (4.15) *is ISS stable with respect to additive disturbances* d_x *and measurement noises* d_y.

Only qualitative analysis of robustness (*i.e.* ISS) is presented in this section. The quantitative one needs further research developments using ideas introduced in [Sanfelice and Praly, 2011], [Prasov and Khalil, 2013], [Menard et al., 2017]. The LMIs presented in the previous section are expected to be useful for obtaining of rather precise ellipsoidal estimates of the observation error in the perturbed case.

4.3 Numerical Simulations

The following two numerical simulations aim to show the main properties of the finite-time and the fixed-time observers. In both examples, an inverted-cart pendulum model will be used. In the first example, the robustness of the FT observer will be studied by applying it to the nonlinear plant and a comparison with a high-gain (HG) observer will be included. The second example will focus on the uniformity w.r.t to initial conditions of the FxT observer. A linearized model will be used such that we can compare the performance of the observers with initial conditions far away from the linearizing equilibrium point.

The state vector is given by $\mathbf{x} = [x, \dot{x}, \theta, \dot{\theta}]^T$, where as usual $(x, \dot{x})$ represents the position and the velocity of the cart and $(\theta, \dot{\theta})$ the angle (from the vertical down position) and the angular velocity of the pendulum. The model parameters are $M = 0.5$ Kg - mass of the cart, $m_c = 0.2$ Kg - mass of the pendulum, $b = 0.1$ N/m/s - cart friction coefficient, $l = 0.3$ m - length to pendulum center of mass, $I = 0.006$ Kg - moment of inertia of the pendulum.

It is assumed that only the cart position and pendulum angle can be measured directly. The nonlinear equations describing the system motion are given by

$$
\begin{aligned}
(M + m_c)\ddot{x} + b\dot{x} + m_c - l\ddot{\theta}\cos\theta - m_c l\dot{\theta}^2 \sin\theta &= F_{\text{in}} \\
(I + m_c l^2)\ddot{\theta} + m_c g l \sin\theta + m_c l \ddot{x}\cos\theta &= 0,
\end{aligned}
$$

where F_{in} represents the input force. A simple proportional control law $F_{\text{in}} = K_p \frac{d}{dt}\hat{\theta}(t)$ is used to stabilize the pendulum around the downward position and $\frac{d}{dt}\hat{\theta}(t)$ is an estimate of the angular velocity, to be obtained using the observers. The linearization of the model around the downward equilibrium point $\mathbf{x}_0 = 0 \in \mathbb{R}^4$ gives the following parameters for (4.1):

$$
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.672 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4545 & -31.181 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1.818 \\ 0 \\ -4.545 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

The first step of the observer design is to transform the linearized model into the observable canonical form, given in Lemma 4.1, by obtaining the matrix $\Phi = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ b & 0 & 0 & 1 \end{smallmatrix}\right)$, where $a = -0.1818$ and $b = -0.4545$.

Example 4.1 *Comparison between FT and HG observers*

In order to make a fair comparison, the parameters of both observers have been adjusted in order to to have a similar time response for the initial condition $x_0 = (0, -2, \pi/4, 1)$; that is to say, the norm of the estimation errors is admitted to be less than 0.15 for $t \geq 0.5$ s. Using

Algorithm 4.1 with $(\mu,\alpha,\tau,\xi) = (0.25, 2.5, 100, 4)$, we design the FT observer (4.4), (4.5) with

$$L_{FT} = \begin{pmatrix} -9.9080 & 0 & 0 & -46.8452 \\ 0 & -9.9080 & -46.8452 & 0 \end{pmatrix}^T, \tilde{P} = 0.0969 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and we compare it with the HG observer, designed according to Remark 4.1. Namely, the term $\|\tilde{P}C_0^{-1}\sigma\|$ in (4.5) is replaced by $\epsilon^{(1+(m-1)\mu)/\mu}$.

Note that, the gain factor $\epsilon = 0.3$ (and correspondingly the admissible estimation error $\|e(0.5)\| \leq 0.15$) is selected sufficiently large since the HG observer becomes more sensitive with respect to measurement noises as ϵ decreases. The estimation error of the FT observer turned out to be 10 times less ($\|e(t)\| \leq 0.01$) for $t \geq 0.5$ s. The numerical simulations have been done using the explicit Euler method with a sampling period of 10^{-4}.

Figure 4.1.a depicts the evolution of the observation errors of FT and HG observers for the noise-free case. Although the observers were designed using a linearized model, the system remains stable for the complete nonlinear model. It is also worth noting that the FT observer demonstrates a smaller peaking during transients.

To compare the observers for the case of noisy measurements, a band-limited white noise of power 10^{-5} has been added to the output signal during the simulation. The corresponding results are presented at Figure 4.1.b. They show almost twice better precision (in both $L^2_{(0.5,1.5)}$ and $L^\infty_{(0.5,1.5)}$ norms of the error) of the FT observer with respect to the HG one. This fact has a simple explanation in the context of high-gain observer theory: *since the gain factor ϵ of the FT observer depends on the available part of the observation error (see Remark 4.1), namely,*

$$\epsilon = \epsilon(\sigma) = \|\, \tilde{P}\, C_0^{-1}\sigma\|^{\mu/(1+(m-1)\mu)}, \; \sigma = y - C\hat{x} = Ce,$$

then its value is automatically adapted to the noises of different magnitude (the larger the noise magnitude, the smaller the gain). In the noisy case the convergence time of the FT observer slightly increases, allowing a better estimation precision.

Example 4.2 ***Uniformity w.r.t. to initial conditions***

Here we compare the FT and the FxT observers assuming that the FT observer is derived as the homogeneous approximation of the FxT observer at zero, *i.e.* $L_{FT} = 0.5L_{FX}$ and $\tilde{P}=\tilde{P}_1$, where

$$L_{FX} = \begin{pmatrix} -3.8624 & 0 & 0 & -6.7081 \\ 0 & -3.8624 & -6.7081 & 0 \end{pmatrix}^T, \; \tilde{P}_1 = \begin{pmatrix} 0.0233 & 0 \\ 0 & 0.0233 \end{pmatrix}, \; \tilde{P}_2 = \begin{pmatrix} 0.2589 & 0 \\ 0 & 0.2589 \end{pmatrix}$$

are the gain matrices of the FxT observer obtained applying Algorithm 4.2 with the parameters $(\mu,\alpha,\tau,\xi,\delta_1,\delta_2,\beta) = (0.12, 0.006, 15, 1, 0.3, 10/3, 0.3)$. The comparison results between the FT and FxT observers in *the noise-free case* are depicted in Figure 4.2. They confirm low convergence time sensitivity with respect to initial conditions for the FxT algorithm; for the

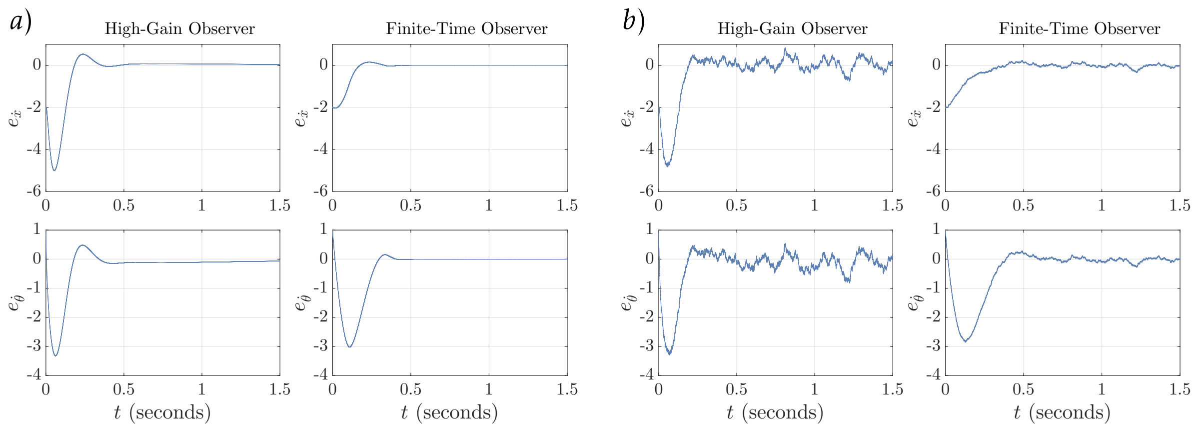

Figure 4.1 – Simulation plots of $e_{\dot{x}}$ and $e_{\dot{\theta}}$ for the HG and FT observers applied to the nonlinear plant with initial conditions $x(0) = (0, -2, \frac{\pi}{4}, 1)$ and $K_p = 10$. *a*) shows the noise-free case and in *b*) a band-limited white noise was introduced in the measurements.

FT one it is possible to see that the convergence time increases drastically as the norm of the initial conditions increases.

The result of the simulations are depicted using a logarithmic scale in order to demonstrate the fast (hyper-exponential) convergence rate of the observers.

Since locally (close to the origin of the error system) the FxT algorithm almost coincides with the FT one, it has almost the same sensitivity with respect to measurement noises (see Fig. 4.3).

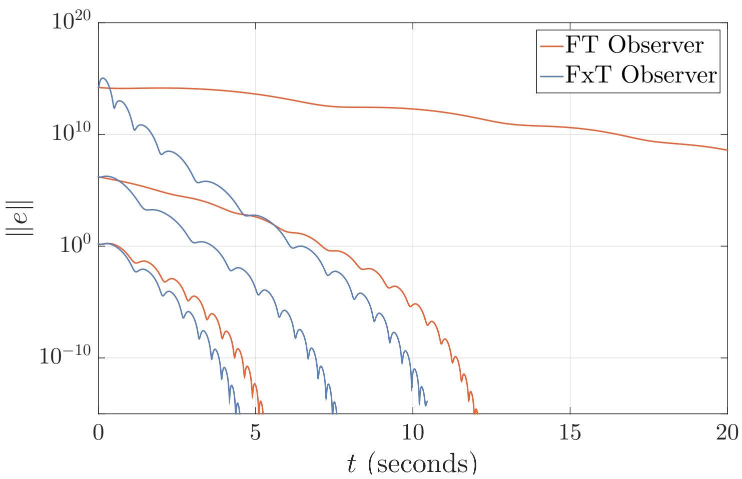

Figure 4.2 – Simulation plot of $\|e\|$ for the FT and FxT observers applied to the linearized plant for three different initial conditions $x(0) = (0, 1, \frac{\pi}{4}, 0), x(0) = 10^3(0, 1, \frac{\pi}{4}, 0), x(0) = 10^7(0, 1, \frac{\pi}{4}, 0)$ and $K_p = 5$.

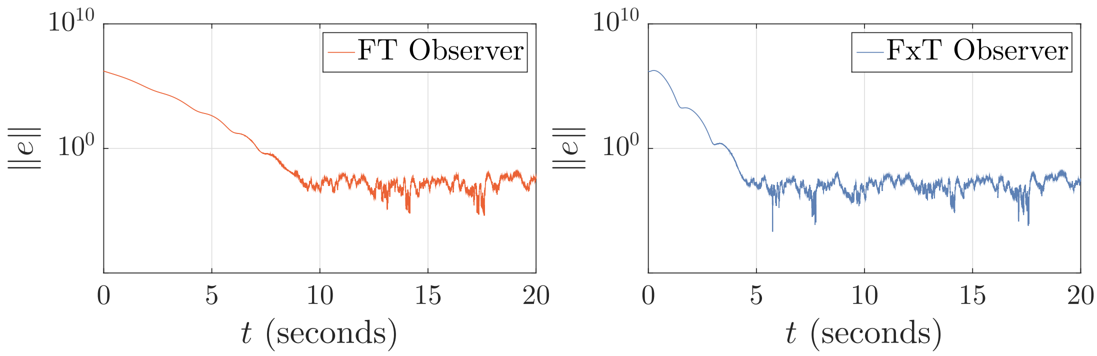

Figure 4.3 – Simulation plot of $||e||$ for the FT and FxT observers with measurement band limited noise in the linearized plant, with initial conditions $x(0) = 10^3(0, 1, \pi, 0)$ and $K_p = 5$.

4.4 Experimental Results

In this section we present a set of experimental results, obtained by applying the FT observer in order to estimate the angular velocities of a rotary pendulum, also known as the Furuta pendulum. The performance of the angular velocity estimation will be compared against a conventional high-pass filter.

The physical setting consists of a QUANSER QUBE-Servo 2 rotary pendulum, depicted in Figure 4.4. The state of the pendulum is described by the vector $x = [\theta, \alpha, \dot{\theta}, \dot{\alpha}]^T$ where θ is the rotary angle and α is the pendulum angle, both measured in radians, $\alpha = 0$ represents the pendulum's downward position and a positive increment of α represents a turn in the counter-clockwise direction (see Figure 4.4).

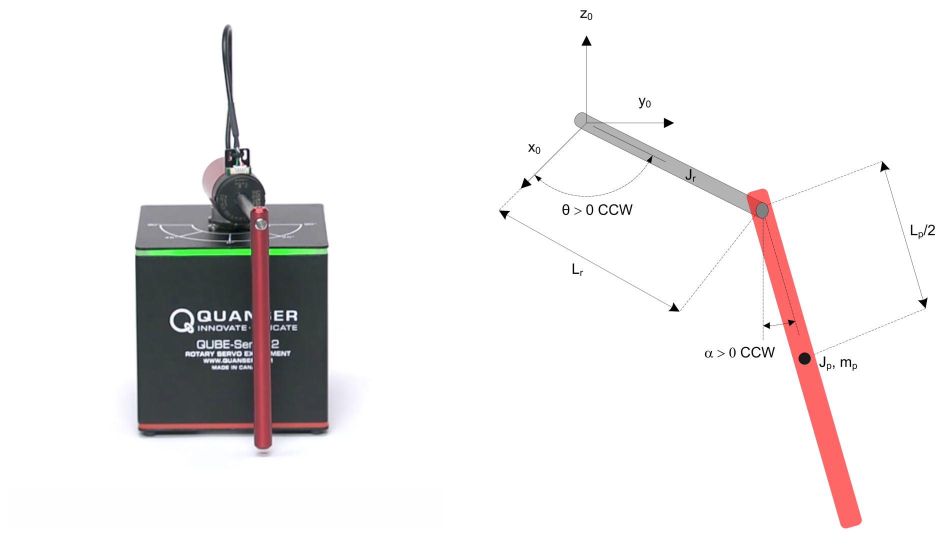

Figure 4.4 – Picture and free body diagram of the QUANSER rotary pendulum.

The nonlinear equations of the rotary pendulum are given by

$$\begin{aligned}\left(m_pL_r^2+\tfrac{1}{4}m_pL_p^2-\tfrac{1}{4}m_pL_p^2\cos(\alpha)^2+J_r\right)\ddot{\theta}-\left(\tfrac{1}{2}m_pL_pL_r\cos(\alpha)\right)\ddot{\alpha}&\\ +\left(\tfrac{1}{2}m_pL_p^2\sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha}+\left(\tfrac{1}{2}m_pL_pL_r\sin(\alpha)\right)\dot{\alpha}^2&=\tau-D_r\dot{\theta} \quad (4.16)\\ \tfrac{1}{2}m_pL_pL_r\cos(\alpha)\ddot{\theta}+\left(J_p+\tfrac{1}{4}m_pL_p^2\right)\ddot{\alpha}&\\ -\tfrac{1}{4}m_pL_p^2\cos(\alpha)\sin(\alpha)\dot{\theta}^2+\tfrac{1}{2}m_pL_pg\sin(\alpha)&=-D_p\dot{\alpha}, \quad (4.17)\end{aligned}$$

where m, L, J and D represent the mass, the length, the moment of inertia and the damping coefficient of either the pendulum (subindex p) or the rotary arm (subindex r).

The torque τ, applied at the base of the rotary arm, is generated by the servo motor as described by the equation

$$\tau=\frac{k_m\left(V_m-k_m\dot{\theta}\right)}{R_m},$$

where k_m is the motor's back-emf constant and R_m is the motor's terminal resistance.

The pendulum's parameter values are as follows:

Rotary DC Motor		
Param.	Value	Units
R_m	8.4	Ω
k_m	0.042	$\frac{V\cdot s}{\text{rad}}$

Rotary Arm		
Param.	Value	Units
m_r	0.095	kg
L_r	0.085	m
J_r	$\frac{m_r\cdot L_r^2}{12}$	$kg\cdot m^2$
D_r	0.001	$\frac{N\cdot m\cdot s}{\text{rad}}$

Pendulum Link		
Param.	Value	Units
m_p	0.024	kg
L_p	0.129	m
J_p	$\frac{m_p\cdot L_p^2}{12}$	$kg\cdot m^2$
D_p	0.00001	$\frac{N\cdot m\cdot s}{\text{rad}}$

The linearization[2] around the downward equilibrium point yields the following values for the state-space model:

$$A=\frac{1}{J_T}\begin{bmatrix}0 & 0 & J_T & 0\\ 0 & 0 & 0 & J_T\\ 0 & \frac{1}{4}m_p^2L_p^2L_rg & -\left(J_p+\frac{1}{4}m_pL_p^2\right)D_r & \frac{1}{2}m_pL_pL_rD_p\\ 0 & -\frac{1}{2}m_pL_pg\left(J_r+m_pL_r^2\right) & \frac{1}{2}m_pL_pL_rD_r & -\left(J_r+m_pL_r^2\right)D_p\end{bmatrix}$$

and

$$B=\frac{1}{J_T}\begin{bmatrix}0 & 0 & J_p+\frac{1}{4}m_pL_p^2 & -\frac{1}{2}m_pL_pL_r\end{bmatrix}^T,\quad C=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix},\quad D=\begin{bmatrix}0\\ 0\end{bmatrix},$$

where $J_T=J_pm_pL_r^2+J_rJ_p+\frac{1}{4}J_rm_pL_p^2$.

The physical platform of the rotary pendulum provides photo sensors that allow to measure both θ and α as increments in the rotary discs and convert them into values in radians. The goal of the experiment is to apply a voltage input signal to the rotational motor and estimate the angular velocities $\dot{\theta}$ and $\dot{\alpha}$ using only measurements of the angles θ and α.

[2] For more details on the pendulum's modeling and its linearization refer to the QUANSER QUBE-Servo 2 user manual (https://www.quanser.com/products/qube-servo-2/).

Two cases are considered: 1) A high-pass filter with transfer function $\frac{50s}{s+50}$ is applied to each of the two angle measurements θ and α in order to estimate $\dot{\theta}$ and $\dot{\alpha}$. This is in fact the method used by the manufacturer to estimate the angular velocities. 2) The finite-time observer (4.4), (4.5) with parameters $(\mu, \alpha, \tau, \xi) = (0.3, 2.5, 100, 4)$.

The selection of the observer parameters has been made following the same procedure as the numerical example in the previous section, this is, the parameters have been chosen for the FT observer to have a similar time responses as the high-pass filtering. The parameter selection of the FT observer yields, using Algorithm 4.1, the gain matrices

$$L_{FT} = \begin{pmatrix} -9.976 & 0 & 0 & 51.674 \\ 0 & -9.976 & 51.674 & 0 \end{pmatrix}^T, \tilde{P} = 0.0139 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

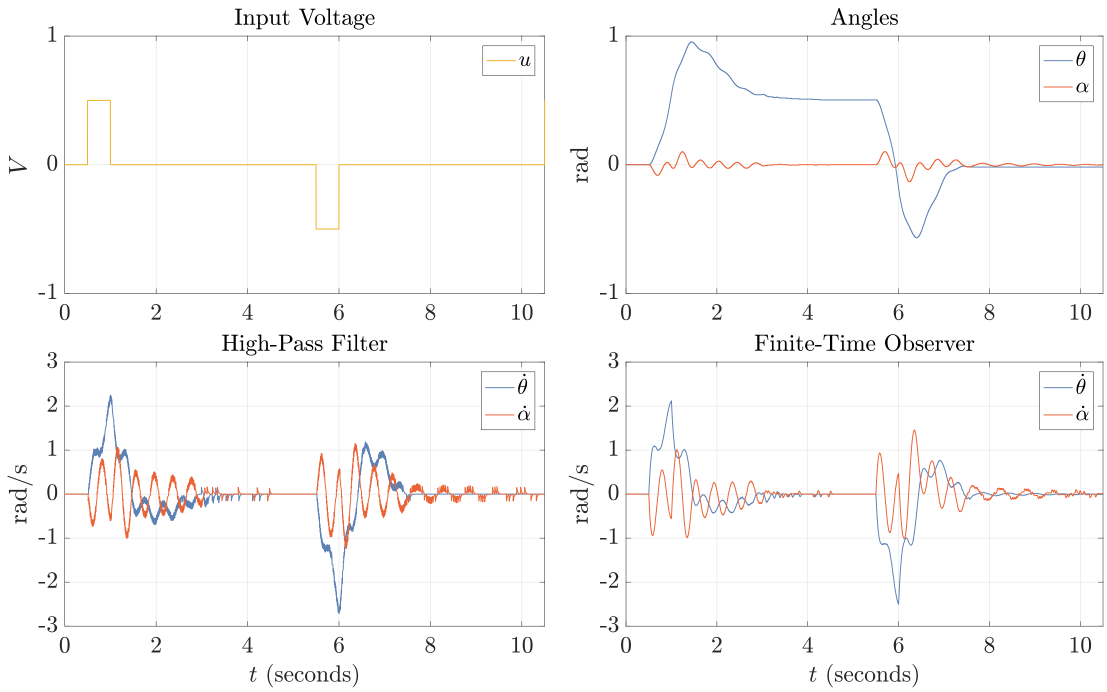

Figure 4.5 – Experiment's results. The upper left figure shows the input voltage signal. The upper right plot shows the angle measurements, obtained from the pendulum's photo sensors. The lower part of the figure shows the estimations of the angular velocities obtained with the high-pass filter (left) and with the FT observer (right).

The experiment's results are depicted in Figure 4.5, they were obtained using the manufacturer's Simulink interface and they confirm the noise reduction property of the FT observer. Note that no additional noise has been added to the physical setting, so that the noise present in the angular velocity measurements is produced exclusively by the quantization errors of the photo sensors. To highlight the noise reduction effect, Figure 4.6 depicts the frequency spectrum of the measurements signals of $\dot{\theta}$ and $\dot{\alpha}$. It is possible to see that the amplitude of the high-frequency components of the measurement signals is significantly smaller in the FT observer.

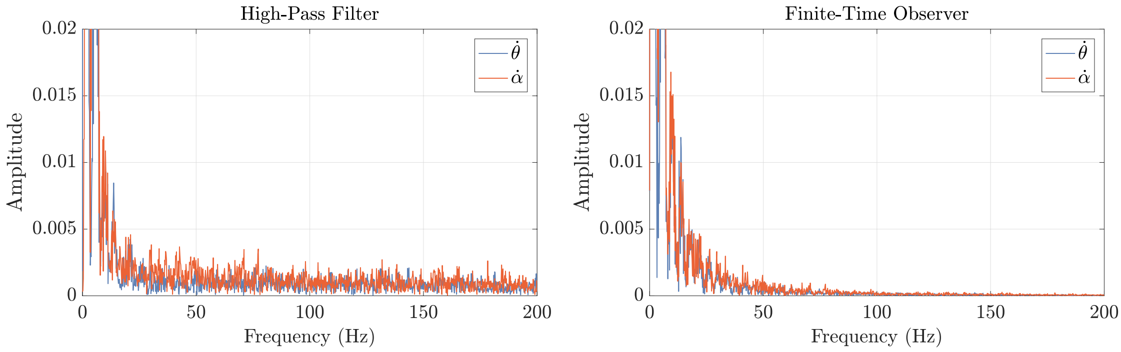

Figure 4.6 – Frequency spectrum of the measurement signals $\dot{\theta}$ and $\dot{\alpha}$ of the high-pass filter (left) and the FT observer (right).

4.5 Conclusions

This chapter presents finite-time and fixed-time nonlinear observers for MIMO linear systems. Their key features are homogeneity properties and the use of the implicit Lyapunov function method for stability analysis of the error equation. The former allows the observers to attain NonA (finite-time or fixed-time) convergence while the latter simplifies the tuning of the observers' gains using LMI-based algorithms. The design is based on a transformation to a canonical observability form so that similar observers can be easily applied to nonlinear systems that admit this canonical form. In the case of the FT observer, the presented algorithm always finds a feasible solution.

The observers' robustness against bounded measurement noises and disturbances was also studied. It was shown that while both observers are ISS stable, the FT observer becomes iISS if the parameter μ is equal to zero.

The performance of the observers was tested through numerical simulations and physical experiments. They confirm a noise reduction effect compared to asymptotic observers. Quantitative robustness analysis (*e.g.* construction of a sharp estimate of the observation error in the perturbed case) is considered an important problem for further research.

4.6 Proofs

Block Decomposition

Let the matrices T_i be defined by the following algorithm:
Initialization : $A_1 = A$, $C_1 = C$, $T_1 = I_n$, $m = 1$.
Loop: While $\mathrm{rank}(C_m) < \mathrm{rown}(A_m)$ do $T_{m+1} = \begin{pmatrix} \hat{C}_m & C_m^{\perp} \end{pmatrix}$, $A_{m+1} = (C_m^{\perp})^T A_m C_m^{\perp}$, $C_{m+1} = \hat{C}_m^T A_m C_m^{\perp}$, $m = m+1$, where $C_m^{\perp} := \mathrm{null}(C_m)$, $\hat{C}_m := \mathrm{null}\left(\left(C_m^{\perp}\right)^T\right)$.

This simple algorithm can be easily realized in MATLAB and it helps to construct an orthogonal coordinate transformation that decomposes the original system (4.1) into a block upper diagonal canonical form. If the pair (A, C) is observable, then the algorithm given above stops after m steps, where $m < n$, and the matrix $\mathcal{O} = T_1 \begin{pmatrix} I_{w_2} & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} I_{w_3} & 0 \\ 0 & T_3 \end{pmatrix} \dots \begin{pmatrix} I_{w_m} & 0 \\ 0 & T_m \end{pmatrix}$, where $w_i := n - \mathrm{rown}(T_i)$, is an orthogonal matrix such that $\mathcal{O}^T\mathcal{O} = \mathcal{O}\mathcal{O}^T = I_n$ and

$$\mathcal{O}A\mathcal{O}^T = \begin{pmatrix} A_{11} & A_{12} & 0 & \dots & 0 \\ A_{21} & A_{22} & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m\text{-}1\,1} & A_{m\text{-}1\,2} & \dots & A_{m\text{-}1\,m\text{-}1} & A_{m\text{-}1\,m} \\ A_{m\,1} & A_{m\,2} & \dots & A_{m\,m-1} & A_{m\,m} \end{pmatrix}, \quad C\mathcal{O} = \begin{pmatrix} C_0 & 0 & \dots & 0 \end{pmatrix},$$

where $C_0 = C\hat{C}_1$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $n_i := \mathrm{rank}(C_i)$, $i,j = 1,\dots,m$ and $\mathrm{rank}(A_{i\,i+1}) = n_{i+1}$. This can be proven, for example, using ideas of duality and Lemma 3 from [Polyakov, 2012].

Since $\mathrm{rank}(A_{i\,i+1}) = n_{i+1} = \mathrm{rown}(A^T_{i\,i+1})$ then $A^T_{i\,i+1} \cdot A_{i\,i+1}$ is invertible and $A^+_{i\,i+1} = (A^T_{i\,i+1}A_{i\,i+1})^{-1}A^T_{i\,i+1}$ is the left inverse matrix to $A_{i\,i+1}$. Consider now the next recursive algorithm in the matrix A.

Initialization: $A^{[m]}_{i\,j} = A_{i\,j}, \quad i,j = 1,2,..,m$

Loop: for $q = m, m-1, \dots, 2$

for $p = 0, 1, \dots, q-2$

for $j = 1, 2, \dots, q-p-1$

$A^{[q\text{-}p\text{-}1]}_{q\,j} = A^{[q\text{-}p]}_{q\,j} - A^{[q\text{-}p]}_{q\,q\text{-}p} \cdot A^+_{q\text{-}p\text{-}1\,q\text{-}p} \cdot A^{[q\text{-}p]}_{q\text{-}p\text{-}1\,j}$

end

$A^{[q\text{-}1]}_{q\text{-}1\,q\text{-}p\text{-}1} = A^{[q]}_{q\text{-}1\,q\text{-}p\text{-}1} + A^{[q]}_{q\text{-}1\,q} \cdot A^{[q\text{-}p]}_{q\,q\text{-}p} \cdot A^+_{q\text{-}p\text{-}1\,q\text{-}p}$

end

end

where the superscript $[m]$ represents the m-th iteration over the matrix A. Then it can be shown that the transformation

$$\Phi = \begin{pmatrix} I_{n_1} & 0 & \dots & 0 & 0 \\ -A^{[2]}_{2\,2}A^+_{1\,2} & I_{n_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A^{[2]}_{m-1\,2}A^+_{1\,2} & -A^{[3]}_{m-1\,3}A^+_{2\,3} & \dots & I_{n_{m-1}} & 0 \\ -A^{[2]}_{m\,2}A^+_{1\,2} & -A^{[3]}_{m\,3}A^+_{2\,3} & \dots & -A^{[m]}_{m\,m}A^+_{m-1\,m} & I_{n_m} \end{pmatrix}\mathcal{O} \tag{4.18}$$

reduces the original matrix A to the block form: $\Phi A \Phi^{-1} = F\tilde{C} + \tilde{A}$, where $n_1 = k$, $F = (A^{[1]}_{11}, A^{[1]}_{21}, \dots, A^{[1]}_{m\text{-}1}, A^{[1]}_{m1})^T$ and

$$\tilde{C} = [I_{n_1}\ 0] \in \mathbb{R}^{n_1 \times n}, \quad \tilde{A} = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{m-1\,m} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Proof of Theorem 4.1

I. Show that the function Q defined by (4.2) satisfies the conditions **C1**-**C3** of Theorem 1.14. It is continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$. Since $P > 0$, then the inequalities

$$\frac{\lambda_{\min}(P)\|z\|^2}{\max\{V^{2\min r_i}, V^{2\max r_i}\}} \le Q(V,z) + 1 \le \frac{\lambda_{\max}(P)\|z\|^2}{\min\{V^{2\min r_i}, V^{2\max r_i}\}}$$

imply that for any $z \in \mathbb{R}^n \backslash \{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+$ such that $Q(V^-, z) < 0 < Q(V^+, z)$.

Moreover, if $Q(V,z)=0$ then, obviously, the condition **C3** of Theorem 1.13 holds. Since $\frac{\partial Q}{\partial V} = -V^{-1}z^T D_r(V^{-1})(H_rP+PH_r)D_r(V^{-1})z$, then $H_rP+PH_r>0$ implies $\frac{\partial Q}{\partial V}<0$ for all $V\in\mathbb{R}_+$ and all $z\in\mathbb{R}^n\backslash\{0\}$. So the condition **C4** of Theorem 1.13 also holds. Therefore, the equation $Q(V,z)=0$ implicitly defines a positive definite Lyapunov function candidate $V:\mathbb{R}^n\to\mathbb{R}$.

II. Let us denote $\lambda=\|\tilde{P}\tilde{C}e\|/V$ and show that $0\le\lambda\le 1$ if (4.8d) holds. By defining $\tilde{e}:=D_r(V^{-1})e$, we have that $\lambda=\|\tilde{P}\tilde{C}e\|/V=\|\tilde{P}\tilde{C}\tilde{e}\|$ and $\tilde{e}^TP\tilde{e}=1$ (due to $Q(V,e)=0$). Given (4.8d) and that $\tilde{P}=X^{1/2}$, we obtain $\lambda^2=\tilde{e}^T\tilde{C}^TX\tilde{C}\tilde{e}\le\tilde{e}^TP\tilde{e}=1$.

III. Denote by ∂Q^e the partial derivative of Q along (4.7):

$$\partial Q^e = 2e^TD_r(V^{-1})PD_r(V^{-1})\left(\tilde{A}+D_{\tilde{r}}\left(\|\tilde{P}\tilde{C}e\|^{-1}\right)L_{FT}\tilde{C}\right)e.$$

Taking into account the identities $D_{\tilde{r}}(V)V^{-1}=D_r(V^{-1})V^{\mu/(1+(m-1)\mu)}$, $D_r(V^{-1})\tilde{A}D_r^{-1}(V^{-1})=V^{\frac{-\mu}{1+(m-1)\mu}}\tilde{A}$ and $D_{\tilde{r}}(V^{-1})L_{FT}\tilde{C}=V^{\frac{-\mu}{1+(m-1)\mu}}D_r^{-1}(V^{-1})L_{FT}\tilde{C}D_r(V^{-1})$ we derive

$$\partial Q^e = V^{\frac{-\mu}{1+(m-1)\mu}}\begin{pmatrix} D_r(V^{-1})e \\ \Xi(\lambda)L_{FT}\frac{\tilde{C}e}{\|\tilde{P}\tilde{C}e\|}\end{pmatrix}^T\Upsilon\begin{pmatrix} D_r(V^{-1})e \\ \Xi(\lambda)L_{FT}\frac{\tilde{C}e}{\|\tilde{P}\tilde{C}e\|}\end{pmatrix}$$

where $\Upsilon=\begin{pmatrix} P(\tilde{A}+L_{FT}\tilde{C})+(\tilde{A}+L_{FT}\tilde{C})^TP & P \\ P & 0\end{pmatrix}$. Using the matrix inequality (4.8a) with $Y=PL_{FT}$ and the identity $e^TD_r(V^{-1})PD_r(V^{-1})e=1$ we estimate

$$\partial Q^e\le V^{\frac{-\mu}{1+(m-1)\mu}}\left(-\alpha e^TD_r(V^{-1})(PH_r+H_rP)D_r(V^{-1})e+\xi\frac{(\tilde{P}\tilde{C}e)^T}{\|\tilde{P}\tilde{C}e\|}\tilde{P}^{-1}L_{FT}^T\Xi(\lambda)Z\Xi(\lambda)L_{FT}\tilde{P}^{-1}\frac{\tilde{P}\tilde{C}e}{\|\tilde{P}\tilde{C}e\|}-\xi\right).$$

Since (4.8c) is equivalent to $\tilde{P}^{-1}L_{FT}^TPL_{FT}\tilde{P}^{-1}\le\tau I_k$ with $\tilde{P}=X^{1/2}$ and taking into account (4.8d) and (4.8f) we derive

$$\partial Q_e\le\frac{-\alpha e^TD_r(V^{-1})(PH_r+H_rP)D_r(V^{-1})e}{V^{\frac{\mu}{1+(m-1)\mu}}}<\alpha V^{1-\frac{\mu}{1+(m-1)\mu}}\frac{\partial Q}{\partial V}.$$

Finally, applying Theorem 1.14 we finish the proof.

Proof of Proposition 4.1

Denote $W(\lambda)=z^T\Xi(\lambda)Z\Xi(\lambda)z$. Since $\Xi'(\theta)=\frac{1}{\theta}\Xi(\theta)(I_n-H_{\tilde{r}})-H_{\tilde{r}}$, then

$$\begin{aligned} W'(\theta) &= z^T(\Xi'(\theta)Z\Xi(\theta)+\Xi(\theta)Z\Xi'(\theta))z \\ &= z^T\begin{pmatrix}\Xi(\theta)\\ I_n\end{pmatrix}^T\begin{pmatrix}\frac{1}{\theta}(I_n-H_{\tilde{r}})Z+\frac{1}{\theta}Z(I_n-H_{\tilde{r}}) & -ZH_{\tilde{r}} \\ -H_{\tilde{r}}Z & \theta M\end{pmatrix}\begin{pmatrix}\Xi(\theta)\\ I_n\end{pmatrix}z-\theta z^TMz\end{aligned}$$

and due to (4.9c) and the Schur complement we have $W'(\theta)\ge-\theta z^TMz$ and $W(\lambda)\le W(q_i)-\frac{1}{2}(\lambda^2-q_i^2)z^TMz$ for any $\lambda\in[q_{i-1},q_i]$, $i=1,\ldots.,N$. Hence, the set of inequalities (4.9) imply (4.8f).

Proof of Theorem 4.2

I. The function Q_i defined by (4.2) with $r=r_i, i=1,2$ satisfies the conditions **C1**-**C4** of Theorem 1.13 (see proof of Theorem 4.1). Note that $Q_1(1,z)=Q_2(1,z)$ for all $z\in\mathbb{R}^n$. In order to complete the proof we need to show that the conditions **C7**-**C9** of Theorem 1.15 hold.

II. Let $\tilde{P}_i = \delta_i \tilde{P}$ with $\tilde{P} = X^{1/2}$. In the proof of Theorem 4.1, it was shown that $0 \leq \lambda_1 = \frac{\|\tilde{P}\tilde{C}e\|}{V} \leq 1$ with $Q(V,e) = 0$ if (4.12e) holds. The same result remains true for $Q_i(V,e) = 0$.

III. Let ∂Q_i^e be the derivative of Q_i along the equation (4.11)

$$\partial Q_i^e = 2e^T D_{r_i}(V^{-1}) P D_{r_i}(V^{-1}) \left(\tilde{A} + \frac{D_{\tilde{r}}\left(\frac{1}{\delta_1 \lambda_1 V}\right) + D_{\tilde{r}}(\delta_2 \lambda_1 V)}{2} L_{FX} \tilde{C} \right) e, \quad \lambda_1 = \frac{\|\tilde{P}\tilde{C}e\|}{V}.$$

Since $D_{r_i}(\rho) = \rho^{1 + \frac{(-1)^{i+1}\mu}{1+(m-1)\mu}} D_{\tilde{r}}\left(\rho^{(-1)^i}\right)$, $\rho > 0$ then

$$\partial Q_i^e = V^{\frac{(-1)^i \mu}{1+(m-1)\mu}} \begin{pmatrix} D_{r_i}(V^{-1})e \\ \bar{\Xi}_i^{\delta}(\lambda) L_{FX} \frac{\tilde{C}e}{\|\tilde{P}\tilde{C}e\|} \end{pmatrix}^T \chi \begin{pmatrix} D_{r_i}(V^{-1})e \\ \bar{\Xi}_i^{\delta}(\lambda) L_{FX} \frac{\tilde{C}e}{\|\tilde{P}\tilde{C}e\|} \end{pmatrix},$$

$\lambda = (\lambda_1, \lambda_2)$, where $\chi = \begin{pmatrix} P(\tilde{A}+L_{FX}\tilde{C}) + (\tilde{A}+L_{FX}\tilde{C})^T P & P \\ P & 0 \end{pmatrix}$, $\lambda_2 = V^2 \in (0,1]$ if $i = 1$ and $\lambda_2 = 1/V^2 \in (0,1]$ if $i = 2$. Repeating the considerations of the proof of Theorem 4.1 we derive that (4.12) imply $\partial Q_i^e \leq -\alpha V^{\frac{(-1)^i \mu}{1+(m-1)\mu}} e^T D_{r_i}(\frac{1}{V})(PH_i + H_i P) D_{r_i}(\frac{1}{V}) e$. Taking into account $\frac{\partial Q_i(V,e)}{\partial V} = -V^{-1} e^T D_{r_i}(V^{-1})(H_i P + PH_i) D_{r_i}(V^{-1}) e$ and LMI (4.12d) we derive that all conditions of Theorem 1.15 hold, so that the error equation (4.11) is fixed-time stable and $T_{\max} \leq \frac{1+(m-1)\mu}{0.5\alpha\mu}$.

Proof of Proposition 4.2

I. Consider the function $W_i(\lambda) = z^T \bar{\Xi}_i^{\delta}(\lambda) Z_i \bar{\Xi}_i^{\delta}(\lambda) z$, where $z \in \mathbb{R}^n$ is an arbitrary non-trivial vector. Since

$$\begin{aligned} \frac{\partial \bar{\Xi}_i^{\delta}}{\partial \lambda_1} &= \frac{1}{\lambda_1} \bar{\Xi}_i^{\delta}(\lambda) + \frac{1}{2}\left(D_{\tilde{r}}\left(\delta_2 \lambda_1 / \lambda_2^{i-2}\right) - D_{\tilde{r}}\left(\lambda_2^{i-1}/(\delta_1 \lambda_1)\right)\right) H_{\tilde{r}} \\ &= \frac{1}{\lambda_1}\left(\bar{\Xi}_i^{\delta}(\lambda)(I_n - H_{\tilde{r}}) + D_{\tilde{r}}(\delta_2 \lambda_1/\lambda_2^{i-2}) H_{\tilde{r}} + H_{\tilde{r}} \right), \end{aligned}$$

then

$$\begin{aligned} \frac{\partial W_i}{\partial \lambda_1} &= z^T \frac{\partial \bar{\Xi}_i}{\partial \lambda_1} Z \bar{\Xi}_i^{\delta} z + z^T \bar{\Xi}_i^{\delta} Z \frac{\partial \bar{\Xi}_i^{\delta}}{\partial \lambda_1} z \\ &= z^T \begin{pmatrix} \bar{\Xi}_i^{\delta}(\lambda) \\ D_{\tilde{r}}\left(\frac{\delta_2 \lambda_1}{\lambda_2^{i-2}}\right) \\ I_n \end{pmatrix}^T \begin{pmatrix} \frac{2Z_i - H_{\tilde{r}} Z_i - Z_i H_{\tilde{r}}}{\lambda_1} & Z_i H_{\tilde{r}} & -Z_i H_{\tilde{r}} \\ H_{\tilde{r}} Z_i & 0 & 0 \\ -H_{\tilde{r}} Z_i & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\Xi}_i^{\delta}(\lambda) \\ D_{\tilde{r}}\left(\frac{\delta_2 \lambda_1}{\lambda_2^{i-2}}\right) \\ I_n \end{pmatrix} z. \end{aligned}$$

Using the inequality (4.13c) we derive $\frac{\partial W_i}{\partial \lambda_1} \geq -\lambda_1 \Psi_i(\lambda) - \lambda_1 z^T S_i z$, where

$$\Psi_i(\lambda) = z^T D_{\tilde{r}}\left(\frac{\delta_2 \lambda_1}{\lambda_2^{i-2}}\right) M_i D_{\tilde{r}}\left(\frac{\delta_2 \lambda_1}{\lambda_2^{i-2}}\right) z.$$

On the other hand, the inequality $M_i H_{\tilde{r}} + H_{\tilde{r}} M_i > 0$ implies the estimates $\frac{\partial \Psi_i(\lambda)}{\partial \lambda_1} > 0$ and $\frac{\partial \Psi_i(\lambda)}{\partial \lambda_2} \geq 0$. Hence we conclude $\frac{\partial W_i}{\partial \lambda_1} \geq -\lambda_1 z^T D_{\tilde{r}}(\delta_2) M_i D_{\tilde{r}}(\delta_2) z - \lambda_1 z^T S_i z$ for $\lambda \in [0,1] \times [0,1]$ and

$W_i(\lambda) \leq W_i(q_j,\lambda_2) + \frac{q_j^2-\lambda_1^2}{2} z^T(D_{\tilde r}(\delta_2)M_i D_{\tilde r}(\delta_2)+S_i)z$ for all $\lambda_1 \in [q_{j-1}, q_j]$.

II. Since $\bar\Xi_i^\delta(\lambda) = \bar\Xi_i^\delta(\tilde\lambda) + \frac{\lambda_1}{2} D_{\tilde r}\left(\lambda_1^{(-1)^{i+1}}\right) D_{\tilde r}(\lambda_2)$ with $\tilde\lambda = (\lambda_1, 0)$, then we derive the identity $W_i(\lambda) = z^T B_i(\lambda_1)^T\ F_i(\lambda_2) B_i(\lambda_1) z$, where $F_i(\lambda_2) = \left(\begin{smallmatrix} D_{\tilde r}(\lambda_2) \\ I_n \end{smallmatrix}\right) Z_i \left(\begin{smallmatrix} D_{\tilde r}(\lambda_2) \\ I_n \end{smallmatrix}\right)^T$ and $B_i(\lambda_1) = \left(\begin{smallmatrix} \frac{\lambda_1}{2} D_{\tilde r}\left((\lambda_1\delta_{3-i})^{(-1)^{i+1}}\right) \\ \bar\Xi_i^\delta(\tilde\lambda) \end{smallmatrix}\right)$. Let us denote $\kappa(\lambda_2) = \ln(\lambda_2 W)$. Since κ has the derivative

$$\frac{d\kappa}{d\lambda_2} = \frac{1}{\lambda_2 W}\left(W + z^T B_i(q_j)^T \left(\begin{smallmatrix} D_{\tilde r}(\lambda_2) & 0 \\ 0 & I_n \end{smallmatrix}\right) \Gamma_i \left(\begin{smallmatrix} D_{\tilde r}(\lambda_2) & 0 \\ 0 & I_n \end{smallmatrix}\right) B_i(q_j) z\right),$$

where $\Gamma_i = \left(\begin{smallmatrix} H_{\tilde r}Z_i + Z_i H_{\tilde r} & H_{\tilde r} Z_i \\ Z_i H_{\tilde r} & 0 \end{smallmatrix}\right)$. Since $\Gamma_i + \left(\begin{smallmatrix} 0 & 0 \\ 0 & U_i \end{smallmatrix}\right) \geq 0$ due to (4.13a), $\frac{d\kappa}{d\lambda_2} \geq \frac{1}{\lambda_2} - ce^{-\kappa(\lambda_2)}$ with $c = z^T \bar\Xi_i^\delta(\tilde\lambda) U_i$ $\bar\Xi_i^\delta(\tilde\lambda) z$. Hence $W(\lambda) \leq W(\lambda_1, p_s) + c\ln(p_s\lambda^{-1}) \leq W(\lambda_1,p_s) + c\ln(\frac{p_s}{p_{s-1}})$ for all $\lambda_2 \in [p_{s-1}, p_s]$. Therefore, LMIs (4.13a-4.13d) imply $W(\lambda) \leq \tau^{-1} z^T P z$ for all $\lambda \in [0,1]\times[p_0,1]$. Finally, it is easy to check that $W_i(\lambda) \leq \tilde W(\lambda) := z^T B_i(\lambda_1)^T \left(\begin{smallmatrix} D_{\tilde r}(\lambda_2) Z_i D_{\tilde r}(\lambda_2) + D_{\tilde r}(\lambda_2) & 0 \\ 0 & Z_i + Z_i D_{\tilde r}(\lambda_2) Z_i \end{smallmatrix}\right) B_i(\lambda_1) z$. Since $Z_i H_{\tilde r} + H_{\tilde r} Z_i \geq 0$ then $\frac{\partial \tilde W}{\partial \lambda_2} \geq 0$ and $W(\lambda) \leq \tilde W(\lambda_1, p_0)$ for all $\lambda_2 \in [0, p_0]$. Therefore, the inequality (4.13e) implies $W(\lambda) \leq \tau^{-1} z^T P z$ for all $\lambda \in [0,1]\times[0,p_0]$.

Proof of Corollary 4.2

Denote with $\tilde f(e,d)$ the right-hand side of (4.14), where $d = (d_x, d_y)$. For $d = 0$ it coincides with the right-hand side of (4.7) and defines an r-homogeneous vector field with degree $\frac{-\mu}{1+(m-1)\mu} < 0$. Taking into account that $D_r^{-1}(\lambda)\tilde A D_r(\lambda) = \lambda^{-\frac{\mu}{1+(m-1)\mu}} \tilde A$ and that $\tilde C D_r(\lambda) e = \lambda^{r_1} \tilde C e$, we derive $\lambda^{\frac{\mu}{1+(m-1)\mu}} D_r^{-1}(\lambda)(\tilde A + D_{\tilde r}(\frac{1}{\|\tilde P \tilde C D_r(\lambda) e\|}) L_{FT} \tilde C) D_r(\lambda) e = (\tilde A + D_{\tilde r}(\|\tilde P \tilde C e\|^{-1}) L_{FT} \tilde C) e$, therefore the error dynamics (4.7) is r-homogeneous of degree $\eta = -\frac{\mu}{1+(m-1)\mu}$. Selecting $\bar r = (\mathbb{1}_k, r - \frac{\mu}{1+(m-1)\mu}\mathbb{1}_n) \in \mathbb{R}^{k+n}$ and using Theorem 1.10, we conclude ISS for system (4.14) for $\mu \in (0,1)$. If $\mu = 1$ then $\bar r_{\min} = 0$ and only iISS can be asserted for (4.14).

Proof of Corollary 4.3

Denote $\tilde f(e,d)$ the right-hand side of (4.15), where $d = (d_x, d_y)$. For $d = 0$ it defines a vector field $f(\cdot) = \tilde f(\cdot, 0)$ that is locally homogeneous at 0 and at $+\infty$, namely, $(r_1, 0, f_0)$-homogeneous with negative degree $\eta_0 = -\frac{\mu}{1+(m-1)\mu}$ and $(r_2, +\infty, f_\infty)$-homogeneous with positive degree $\eta_\infty = \frac{\mu}{1+(m-1)\mu}$, $f_0 = (\tilde A + \frac{1}{2} D_{\tilde r}(\|\tilde P_1 \tilde C e\|^{-1}) L_{FX} \tilde C) e$ and $f_\infty = (\tilde A + \frac{1}{2} D_{\tilde r}(\|\tilde P_2 \tilde C e\|) L_{FX} \tilde C) e$.

Indeed,

$$\begin{aligned} &\lim_{\lambda\to 0} \lambda^{-\eta} D_{r_1}^{-1}(\lambda)\Big[\frac{D_{\tilde r}(\|\tilde P_1 \tilde C D_{r_1}(\lambda) e\|^{-1}) + D_{\tilde r}(\|\tilde P_2 \tilde C D_{r_1}(\lambda) e\|)}{2} L_{FX} \tilde C D_{r_1}(\lambda) e + \tilde A D_{r_1}(\lambda) e\Big] \\ &= \lim_{\lambda\to 0} (Ae + \frac{1}{2} D_{\tilde r}(\|\tilde P_1 \tilde C e\|^{-1}) L_{FX} \tilde C e) \end{aligned}$$

for $\eta = -\frac{\mu}{1+(m-1)\mu} < 0$, so that the r_1-homogeneous approximation of degree η around 0 of (4.11) is f_0. Analogously, it can be shown that f_∞ is the r_2-homogeneous approximation of (4.11) with degree $\eta_\infty = \frac{\mu}{1+(m-1)\mu} > 0$ at $+\infty$. It is worth stressing that if all the conditions of Theorem 4.2 hold, then the origins of $\dot e = f_0(e)$, $\dot e = f(e)$ and $\dot e = f_\infty(e)$ are globally asymptotically stable. Hence, selecting $\bar{r}^0 = (\mathbb{1}_k, r_1 - \frac{\mu}{1+(m-1)\mu}\mathbb{1}_n) \in \mathbb{R}^{k+n}$ and $\bar{r}^\infty = (\mathbb{1}_k, r_2 + \frac{\mu}{1+(m-1)\mu}\mathbb{1}_n) \in \mathbb{R}^{k+n}$, and using Theorem 1.11 we derive that the system (4.15) is ISS.

Chapter 5

Output Fixed-Time Stabilization of a Chain of Integrators

Contents

This chapter presents an output control scheme that estimates and regulates to zero, both in fixed-time, a chain of integrators. The output control scheme relies in a switching strategy that commutes, from positive to negative, the homogeneity degree of the full control system. As the switch triggering signal, the norm of the states will be used for the controller, whereas an auxiliary dynamics will be introduced for the observer. In the first part of the chapter, rather simple conditions to achieve FxTS of the control setting will be given, however, under these preliminary conditions, it will not be possible to obtain settling-time estimates, nor to determine how the parameter choice will influence the convergence time. In Section 5.2, using the implicit Lyapunov approach, a parameter tuning algorithm that allows to influence the settling-time will be presented.

Consider a chain of integrators:

$$\begin{aligned} \dot{x} &= A_0 x + bu + d, \\ y &= Cx + v, \end{aligned} \qquad t \geq 0, \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measured output; $d \in \mathbb{R}^n$ and $v \in \mathbb{R}$ are time-dependent signals that represent, respectively, the exogenous disturbance and the measurement noise, $d, v \in \mathcal{L}_\infty$ and the matrices

$$A_0 = \begin{pmatrix} 0 & 1 & 0\dots 0 & 0 \\ 0 & 0 & 1\dots 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\dots 0 & 1 \\ 0 & 0 & 0\dots 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad C = (1\ 0\dots 0),$$

are in the upper-diagonal canonical form.

It is required to design a stabilizing dynamic output control $u(t)$ that ensures the ISpS property (see Section 1.2) of the system (5.1) for any $(d,v) \in \mathcal{L}_\infty$; and that for $d = v = 0$ provides global fixed-time stability of the closed-loop system at the origin.

5.1 Output Feedback Control Design

The solution of the problem is divided in three steps. First, a state feedback controller is proposed to ensure the problem solution. Second, the equations of the observer are introduced. Third, a combined output feedback is presented and analyzed.

State feedback

For $i = \overline{1,n}$ and $x_i \in \mathbb{R}$ and $\alpha > 0$ the controller proposed has the form:

$$u(x) = \sum_{i=1}^{n} a_i \lceil x_i \rfloor^{\alpha_i(\nu(\|x\|))}, \quad \alpha_i(\nu) = \frac{1+n\nu}{1+(i-1)\nu}, \tag{5.2}$$

$$\nu(\omega) = \begin{cases} \nu_1 & \text{if } \omega \le m, \\ \nu_2 & \text{if } \omega \ge M, \\ \frac{\nu_2-\nu_1}{M-m}\omega + \frac{M\nu_1 - m\nu_2}{M-m} & \text{otherwise,} \end{cases} \tag{5.3}$$

where $a = (a_1, \dots, a_n) \in \mathbb{R}^{1\times n}$ is the vector of control coefficients forming a Hurwitz polynomial, $-\infty < \nu_1 < 0 < \nu_2 < +\infty$ and $0 < m < M < +\infty$ are the tuning parameters to be defined later. Denote

$$r_i(\nu) = 1 + (i-1)\nu, \quad i = \overline{1,n}, \tag{5.4}$$

then it is straightforward to verify that for $d = 0$ the system (5.1), (5.2) is $r(\nu_1)$–homogeneous of degree $\nu_1 < 0$ for $\|x\| \le m$ and $r(\nu_2)$–homogeneous of degree $\nu_2 > 0$ for $\|x\| \ge M$. Let us show that for properly selected control parameters, the system (5.1), (5.2) is globally fixed-time stable at the origin.

Lemma 5.1. *Let $a \in \mathbb{R}^n$ form a Hurwitz polynomial, then for any $0 < m < M < +\infty$ there exists $\tau \in (0, n^{-1})$ such that if $\nu_1 \in (-\tau, 0)$ and $\nu_2 \in (0,\tau)$ then the system* (5.1), (5.2) *for $d = 0$ is globally fixed-time stable at the origin.*

Proof. Denote

$$A = A_0 + ba = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_1 & a_3 & a_3 & \dots & a_{n-1} & a_n \end{pmatrix},$$

then by the lemma conditions there are matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$P = P^T > 0,\ Q = Q^T > 0,\ A^T P + PA = -Q.$$

Consider for (5.1), (5.2) the Lyapunov function candidate

$$V(x) = x^T P x,$$

whose derivative admits the differential equation:

$$\dot{V}(x) = DV(x)[A_0 x + bu(x)] = -x^T Q x + 2x^T P b\delta(x),$$

where $\delta(x) = \sum_{i=1}^{n} a_i(\lceil x_i \rfloor^{\alpha_i(\nu(\|x\|))} - x_i)$. By construction, $\nu(\|x\|) = 0$ for $\|x\| = \mu = \frac{m\nu_2 - M\nu_1}{\nu_2 - \nu_1}$. Since $\delta : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\nu : \mathbb{R}_+ \to [\nu_1, \nu_2]$, it is possible to select the values of ν_1 and ν_2 sufficiently close to zero such that $|\delta(x)|$ stays small enough, and therefore $\dot{V}(x) < 0$ on any compact containing the level $\|x\| = \mu$. Thus, there exists some $\tau \in (0, n^{-1})$ (if $\tau \geq n^{-1}$ then $\alpha_i(\nu(\|x\|))$ may become non-positive) such that with $\nu_1 \in (-\tau, 0)$ and $\nu_2 \in (0, \tau)$ for all $x \in \{x \in \mathbb{R}^n : \underline{m} \leq \|x\| \leq \overline{M}\}$ we have $\dot{V}(x) < 0$ for any selection of $0 < \underline{m} < m < M < \overline{M} < +\infty$. Using Lemma 1.5, we can prove in this case that the system is $r(\nu_1)$–homogeneous of degree $\nu_1 < 0$ and fixed-time stable at the origin from $\mathcal{B}_{r(\nu_1)}(\rho_1)$ for $\mathcal{B}_{r(\nu_1)}(\rho_1) \subset \{x \in \mathbb{R}^n : \|x\| \leq m\}$, and it is $r(\nu_2)$–homogeneous of degree $\nu_2 > 0$ and globally fixed-time stable with respect to any ball $\mathcal{B}_{r(\nu_2)}(\rho_2)$ such that $\{x \in \mathbb{R}^n : \|x\| = M\} \subset \mathcal{B}_{r(\nu_2)}(\rho_2)$. The constants $\tau, \underline{m}, \overline{M}$ can be selected in a way that $\mathcal{B}_{r(\nu_1)}(\rho_1) \subset \{x \in \mathbb{R}^n : \underline{m} \leq \|x\| \leq m\}$ and $\mathcal{B}_{r(\nu_2)}(\rho_2) \subset \{x \in \mathbb{R}^n : M \leq \|x\| \leq \overline{M}\}$, then (5.1), (5.2) is globally convergent and it is globally fixed-time stable at the origin (the time that the system spent in the set $\{x \in \mathbb{R}^n : \underline{m} \leq \|x\| \leq \overline{M}\}$ is finite). ∎

In order to analyze robust stability properties of the closed loop dynamics (5.1), (5.2) let us introduce the system

$$f_\nu(x, \tilde{d}) := A_0 x + b \sum_{i=1}^{n} a_i \left\lceil x_i + \tilde{d}_{1,i} \right\rfloor^{\alpha_i(\nu)} + \tilde{d}_2,$$

where $\tilde{d} = [\tilde{d}_1^T\ \tilde{d}_2^T]^T \in \mathbb{R}^{2n}$ is the new disturbance input, $\tilde{d}_1$ represents measurements noises and $\tilde{d}_2 = d$.

Corollary 5.1. *Let all conditions of Lemma 5.1 be satisfied, then the system* (5.1), (5.2) *is ISpS for any* $\tilde{d} \in \mathcal{L}_\infty$.

Proof. Consider the system (5.1), (5.2) for $\|x\| \geq M$, and $\dot{x} = f_{\nu_2}(x, \tilde{d})$ is the corresponding approximating system, which is $r(\nu_2)$–homogeneous of degree $\nu_2 > 0$ and globally fixed-time stable with respect to any ball $\mathcal{B}_{r(\nu_2)}(\rho)$ with $\rho > 0$ for $\tilde{d} = 0$. Take $\tilde{r} = \begin{bmatrix} r(\nu_2) \\ r(\nu_2) + \nu_2 \end{bmatrix}$, then $f_{\nu_2}(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)\tilde{d}) = \lambda^\nu \Lambda_r(\lambda) f_{\nu_2}(x, \tilde{d})$ for all $x \in \mathbb{R}^n$, $\tilde{d} \in \mathbb{R}^{2n}$ and all $\lambda > 0$. Consequently, if all conditions of Lemma 5.1 are satisfied, then also all conditions of Theorem 1.10 are true and the system $\dot{x} = f_{\nu_2}(x, \tilde{d})$ is ISS with respect to $d \in \mathcal{L}_\infty$. Since $\dot{x} = f_{\nu_2}(x, \tilde{d})$ is the approximation of (5.1), (5.2) for $\|x\| \geq M$, then (5.1), (5.2) (the system $\dot{x} = f_{\nu(\|x\|)}(x, \tilde{d})$) is ISpS. ■

Thus, the presented state control (5.2) solves the posed problem of robust global fixed-time stabilization for the system (5.1).

Remark 5.1. Clearly, for any $R \in \mathbb{R}^{n\times n}, R = R^T > 0$ the system (5.1), (5.2) with $\nu(\|x\|_R)$, where $\|x\|_R = \sqrt{x^T R x}$, possesses the same properties as (5.1), (5.2) with $\nu(\|x\|)$.

State observer

To explain the observer structure, let us first consider the case $d = v = 0$, then the proposed observer takes the form (see also [Angulo et al., 2013; Cruz-Zavala et al., 2011; Ríos and Teel, 2016]):

$$\begin{aligned} \dot{z}(t) &= A_0 z(t) + bu(t) + k(\nu(\zeta(t)), y(t) - Cz(t)), \\ k_i(\nu, e) &= L_i \lceil e \rfloor^{\beta_i(\nu)},\ \beta_i(\nu) = 1 + i\nu \quad i = \overline{1, n}, \\ \dot{\zeta}(t) &= -0.5\zeta(t) + p(\nu(\zeta(t)), y(t) - Cz(t)), \\ p(\nu, e) &= 4\kappa^T(\nu, e) P \kappa(\nu, e),\ \kappa(\nu, e) = Le - k(\nu, e), \end{aligned} \tag{5.5}$$

where $z(t) \in \mathbb{R}^n$ is the state estimate, $\zeta(t) \in \mathbb{R}_+$ is an auxiliary time function; the function ν is given in (5.3) with $-\infty < \nu_1 < 0 < \nu_2 < +\infty$ and $0 < m < M < +\infty$, are, as previously, the tuning parameters; $L = [L_1, \dots, L_n]^T$ is the vector of coefficients of the observer providing the Hurwitz property of the matrix $A_0 - LC$; $P \in \mathbb{R}^{n\times n}$ is a matrix solution of the equations

$$P = P^T > 0, \quad (A_0 - LC)^T P + P(A_0 - LC) = -P.$$

In [Angulo et al., 2013], instead of using an auxiliary ζ-filter to commute the right-hand sides of (5.5) with negative and positive homogeneity degree, a time switching between two systems with positive and negative homogeneity degree is proposed. In [Ríos and Teel, 2016], in order to switch between observers with negative and positive homogeneity degrees, an hysteresis mechanism is used.

Lemma 5.2. *Let $A_0 - LC$ be a Hurwitz matrix for a given $L \in \mathbb{R}^{n\times 1}$ and assume that the solutions of* (5.1) *are defined for all $t \geq 0$, then for any $0 < m < M < +\infty$ there exists $\tau \in (0, n^{-1})$ such that if $\nu_1 \in (-\tau, 0)$ and $\nu_2 \in (0, \tau)$ then the system* (5.1), (5.5) *for $d = v = 0$ is globally Lyapunov stable and fixed-time convergent with respect to the set $\mathcal{A} = \{(x, z, \zeta) \in \mathbb{R}^{2n+1} : x = z, \|\zeta\| \leq M\}$ for all $(x, z) \in \mathbb{R}^{2n}$, provided that $\zeta(0) > M$ is sufficiently big.*

A more precise restriction on the value of $\zeta(0)$ is given in the proof of this lemma, it is not related with the initial conditions $x(0)$, $z(0)$ (see (5.8)).

Proof. Denote $e = x - z$ as the estimation error, then

$$\begin{aligned}\dot{e} &= A_0 e - k(\nu(\zeta), Ce + v) + d \\ &= (A_0 - LC)e - k(\nu(\zeta), Ce + v) + L(Ce + v) + d - Lv \\ &= (A_0 - LC)e + \kappa(\nu(\zeta), Ce + v) + d - Lv.\end{aligned}$$

Consider a Lyapunov function $V(e) = e^T Pe$, then

$$\begin{aligned}\dot{V} &= -V + 2e^T P[\kappa(\nu(\zeta), Ce + v) + d - Lv] \\ &\leq -0.5V + p(\nu(\zeta), Ce + v) + 4(d - Lv)^T P(d - Lv).\end{aligned}$$

For an auxiliary error variable $\xi = V - \zeta$ we obtain:

$$\dot{\xi} \leq -0.5\xi + 4(d - Lv)^T P(d - Lv)$$

and ξ is exponentially converging to zero (ζ is converging to V) if $d = v = 0$. In addition, if there is an instant of time $t' \geq 0$ such that $\xi(t') \geq 0$, then $\xi(t) \geq 0$ for all $t \geq t'$.

Repeating the arguments of Lemma 5.1, for any $0 < m < M < +\infty$ there exists $\tau \in (0, n^{-1})$ such that if $\nu_1 \in (-\tau, 0)$ and $\nu_2 \in (0, \tau)$ then the system

$$\dot{e} = A_0 e - k(\nu, Ce) \tag{5.6}$$

is globally asymptotically stable for any fixed value of $\nu \in [\nu_1, \nu_2]$. Moreover, for $\nu = \nu_1$ it is $r(\nu_1)$–homogeneous of degree $\nu_1 < 0$ and globally finite-time stable at the origin, and for $\nu = \nu_2$ the system (5.6) is $r(\nu_2)$–homogeneous of degree $\nu_2 > 0$ and globally fixed-time stable with respect to any ball $\mathcal{B}_{r(\nu_2)}(\rho)$ with $\rho > 0$. In addition,

$$\dot{V}(e) < 0 \quad \forall e \in \{e \in \mathbb{R}^n : \underline{m} \leq V(e) \leq \overline{M}\} \tag{5.7}$$

for any selection of $0 < \underline{m} < m < M < \overline{M} < +\infty$ and any (possibly time-varying) value of $\nu(\zeta(t)) \in [\nu_1, \nu_2]$.

Denote by $T_M > 0$ the uniform settling-time of convergence to the ball $\mathcal{B}_{r(\nu_2)}(\rho_M)$ of (5.6) for $\nu = \nu_2$, where $\rho_M > 0$ is such that $\{e \in \mathbb{R}^n : V(e) \leq M\} \subset B_{r(\nu_2)}(\rho_M)$. Let

$$T_M < 2\ln(\zeta(0) - M), \tag{5.8}$$

then $\zeta(t) \geq M$ for $t \in [0, t_M)$ with $t_M \geq T_M$ (t_M can also be infinite), therefore $\nu(\zeta(t)) = \nu_2$ for $t \in [0, t_M)$ from (5.3) and the estimation error dynamics is $r(\nu_2)$–homogeneous and fixed-time stable with respect to the ball $\mathcal{B}_{r(\nu_2)}(\rho_M)$ on this interval of time. Since $t_M \geq T_M$, then the system (5.6) with $\nu = \nu(\zeta(t))$ enters in the ball $\mathcal{B}_{r(\nu_2)}(\rho_M)$ and there is an instant of time $t' \in [0, t_M)$ such that $\xi(t) \geq 0$ for all $t \geq t'$ (*i.e.* $\zeta(t) \geq V(e(t))$ for all $t \geq t'$). Next, due to (5.7), the system (5.6) with $\nu = \nu(\zeta(t))$ reaches the set $\{e \in \mathbb{R}^n : V(e) \leq \overline{m}\}$ in a finite time $T_m > T_M$, where it stays for all $t \geq T_m$. By the properties of the dynamics of ξ an ζ, the instant $t_M < +\infty$ and there is another time instant $t_m \geq \max\{t', T_m\}$ such that $\zeta(t) \leq m$ for all $t \geq t_m$ ($\zeta(t)$ is exponentially approaching $V(e(t))$ from above, while $V(e(t)) \leq \overline{m} < m$ for $t \geq T_m$). Consequently, $\nu(\zeta(t)) = \nu_1$ for $t \geq t_m$ and it reaches for the origin in a uniform time. Summarizing the arguments we obtain that the system (5.6) with $\nu = \nu(\zeta(t))$ is globally fixed-time stable at the origin if $d = v = 0$. The variable ζ is also bounded and exponentially converging to zero. ■

Corollary 5.2. *Let all conditions of Lemma 5.2 be satisfied, then the system* (5.1), (5.5) *is ISpS with respect to the set $\mathcal{A}$ for any $(d, v) \in \mathcal{L}_\infty$.*

Proof. Denote, for brevity, $|\cdot|_\infty = \|\cdot\|_{[0,\infty)}$. From the equation for ξ we have that:

$$V(t) \leq \zeta(t) + (V(0) - \zeta(0))e^{-0.5t} + 8\chi(|v|_\infty, |d|_\infty),$$

where $\sup_{t\geq 0}(d(t) - Lv(t))^T P(d(t) - Lv(t)) \leq \chi(|v|_\infty, |d|_\infty) = \|P\|_2(|d|_\infty^2 + \|L\|_2^2 |v|_\infty^2)$ and $\|\cdot\|_2$ is the induced matrix norm. Consider the set $\Upsilon_0 = \{(e, \zeta) \in \mathbb{R}^{n+1} : V(e) \geq M + \max\{0, V(0) - \zeta(0)\} + 8\chi(|v|_\infty, |d|_\infty)\}$. From the inequality above $\zeta(t) \geq M$ for $e(t) \in \Upsilon_0$. Thus, $\nu(t) = \nu_2$ if $e(t) \in \Upsilon_0$ and the estimation error dynamics takes the form:

$$\dot{e} = f(e, \tilde{d}) = A_0 e - k\left(\nu_2, Ce + \tilde{d}_1\right) + \tilde{d}_2.$$

Denote $\tilde{r} = \begin{bmatrix} r_1(\nu_2) \\ r(\nu_2) + \nu_2 \end{bmatrix}$, then $f(\Lambda_r(\lambda)e, \Lambda_{\tilde{r}}(\lambda)\tilde{d}) = \lambda^\nu \Lambda_r(\lambda) f(x, \tilde{d})$ for all $x \in \mathbb{R}^n$, $\tilde{d} \in \mathbb{R}^{n+1}$ and all $\lambda > 0$. Consequently, if all conditions of Lemma 5.2 are satisfied (the system $\dot{e} = f(e, 0)$ corresponds to (5.6)), then also all conditions of Theorem 1.10 are true and the system $\dot{e} = f(e, \tilde{d})$ is ISS with respect to $\tilde{d} \in \mathcal{L}_\infty$. Therefore, there exists an ISS Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}_+$ (an alternative choice of W can be found in [Bernuau et al., 2013]) and some function ρ of class $\mathcal{K}_\infty$ such that $W(e) \geq \rho(M + \max\{0, V(0) - \zeta(0)\} + 8\chi(|v|_\infty, |d|_\infty))$ implies that $e \in \Upsilon_0$. Then either an ISS estimate holds for W or $W(e) \leq \rho(M + \max\{0, V(0) - \zeta(0)\} + 8\chi(|v|_\infty, |d|_\infty))$, which implies boundedness of W, and the same property for ζ. Consequently, all solutions of (5.5) are defined for all $t \geq 0$ ($x(t)$ is also defined for all $t \geq 0$ by conditions of Lemma 5.2). Then the term $(V(0) - \zeta(0))e^{-0.5t}$ can be skipped and the above consideration can be repeated for the set $\Upsilon = \{(e, \zeta) \in \mathbb{R}^{n+1} : V(e) \geq M + 8\chi(|v|_\infty, |d|_\infty)\}$ in order to prove ISpS with respect to the set $\mathcal{A}$ of the system (5.1), (5.5). Obviously, the variable $\zeta(t)$ is also asymptotically bounded by $M + 16\chi(|v|_\infty, |d|_\infty)$ in this case. ■

Output feedback

The proposed dynamic output feedback consists in the application of the state feedback (5.2) with the state estimates generated by the observer (5.5):

$$u(z)=\sum_{i=1}^{n} a_i \lceil z_i \rfloor^{\alpha_i(\nu(\|z\|))}, \tag{5.9}$$

then the dynamics of the closed-loop system (5.1), (5.5), (5.9) can be written in the coordinates x, $e=x-z$ and ζ as follows:

$$\begin{aligned} \dot{x} &= A_0 x + b\sum_{i=1}^{n} a_i \lceil x_i - e_i \rfloor^{\alpha_i(\nu(\|x-e\|))} + d, \\ \dot{e} &= A_0 e - k(\nu(\zeta), Ce+v) + d, \\ \dot{\zeta} &= -0.5\zeta + p(\nu(\zeta), Ce+v). \end{aligned} \tag{5.10}$$

The main result is a direct consequence of Lemmas 5.1 and 5.2, and Corollaries 5.1 and 5.2.

Theorem 5.1. *Let the following conditions be satisfied:*

i) $a\in\mathbb{R}^{1\times n}$ *forms a Hurwitz polynomial;*

ii) A_0-LC *is a Hurwitz matrix for given* $L\in\mathbb{R}^n$*;*

iii) $\zeta(0)>M$ *is sufficiently big.*

Then for any $0<m<M<+\infty$ *there exists some* $\tau\in(0,n^{-1})$ *such that for* $\nu_1\in(-\tau,0)$ *and* $\nu_2\in(0,\tau)$ *the system* (5.1), (5.5), (5.9) *is*

1) *fixed-time converging with respect to the set* $\{(x,z,\zeta)\in\mathbb{R}^{2n+1} : x=z=0\}$ *for* $d=v=0$ *and for any initial conditions* $(x(0),z(0))\in\mathbb{R}^{2n}$*,*

2) *ISpS for any* $(d,v)\in\mathcal{L}_\infty$*.*

Proof. The system (5.10) is a cascade of the (e,ζ)- and x-dynamics. If $d=v=0$ then (e,ζ)-subsystem is autonomous and globally fixed-time converging with respect to the set $\{(e,\zeta)\in\mathbb{R}^{n+1} : e=0\}$ (with the uniform settling-time $T_o>0$) according to Lemma 5.2. During the interval $[0,T_o]$ the system (5.1) has bounded trajectories due to the ISpS property with respect to measurement noises (estimation errors e) established in Corollary 5.1, and for $t\geq T_o$ the x-subsystem is also autonomous and globally fixed-time converging at the origin by Lemma 5.1.

For $(d,v)\in\mathcal{L}_\infty$ the ISpS property follows the results of Corollaries 5.1, 5.2 and the cascade structure of (5.10). ■

In Theorem 5.1, the same parameters m, M, ν_1 and ν_2 have been selected for the controller (5.9) and for the observer (5.5) in order to keep the notation compact, however, they can be chosen differently in applications and the result of Theorem 5.1 stays correct.

Example 5.1

Let $n = 3$, $L = \begin{bmatrix} 1.5 & 1.01 & 0.25 \end{bmatrix}^T$, $a = -2.5[1\,1\,1]$, $m = 1$, $M = 5$, $\nu_2 = -\nu_1 = 0.1$, $\zeta(0) = 5M$ and

$$P = \begin{bmatrix} 0.121 & -0.13 & 0.047 \\ -0.13 & 0.261 & -0.308 \\ 0.047 & -0.308 & 0.617 \end{bmatrix},$$

then all conditions of Theorem 5.1 are satisfied. We test first the system without perturbations; the results are depicted in the upper left plot of Figure 5.1, where the initial conditions of the system are $x_0 = (5, 10, 0)$ and those of the observer are $z_0 = (0, 0, 0)$. The solid color lines represent the actual state x while the doted color lines represent the estimated state z. It can be seen how the estimated states converge rapidly to the actual states before converging both to zero. In the lower left part of Figure 5.1 we can appreciate the elements of the control scheme, the upper and lower limits of the homogeneity degree M and n are shown as straight lines. The norm of the observed states $\|z\|$ is depicted in yellow, while this norm is between M and m the control's degree of homogeneity lies over the line $\frac{\nu_2-\nu_1}{M-m}\|z\| + \frac{M\nu_1 - m\nu_2}{M-m}$. In the case of the observer, the filter $\zeta(t)$ acts as the modulator of the observer's homogeneity degree. The control signal is shown in green.
Figure 5.1 (right) shows the same setup with initial conditions $x_0 = 10^3(5, 10, 0)$, it can be seen that although the initial state is significantly larger, the settling time remains within the same interval, showing the expected uniformity w.r.t the initial state.
In the lower right part of the figure it can be seen that the control signal u grows considerably to cope with the conditions imposed. We next go back to the previous initial settings and introduce in the control scheme the disturbance

$$d(t) = \sin(2t) + \begin{cases} 10 & \text{if } t \in [30, 31] \\ 0 & \text{otherwise} \end{cases}. \tag{5.11}$$

The results are shown in Figure 5.2. We can notice that the system is robust against this disturbance and its effect in the control scheme elements are depicted in the lower plot of this figure. In particular we can see that the disturbance modifies both $\|z\|$ and $\zeta(t)$ therefore changing the homogeneity degree of both the controller and the observer.

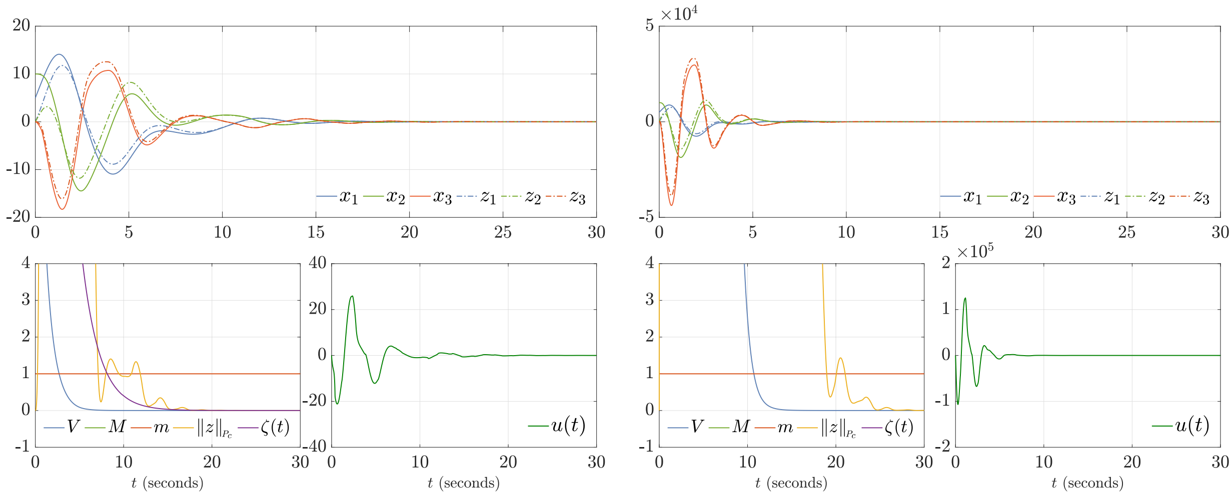

Figure 5.1 – Simulation plot of system (5.1), (5.5) and (5.9) for $n = 3$ without disturbances. On the left-hand side with conditions $x_0 = (5, 10, 0)$, on the right-hand side with initial conditions $x_0 = 10^3(5, 10, 0)$.

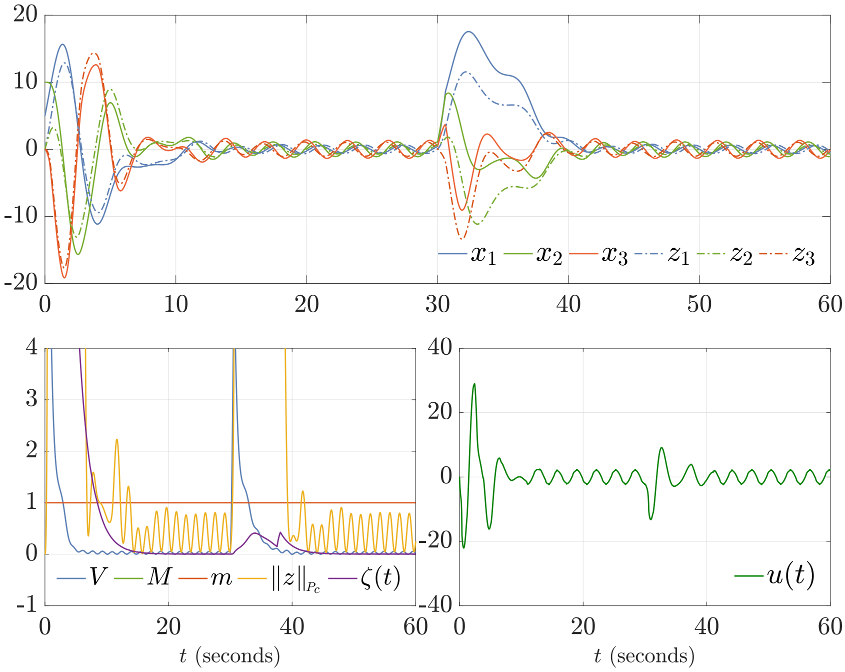

Figure 5.2 – Simulation plot of system (5.1), (5.5) and (5.9) for $n = 3$ with initial conditions $x_0 = (5, 10, 0)$ and the disturbance (5.11).

Remark 5.2. Since in Theorem 5.1, the ISpS property with respect to d is proven, then considering d as a function of x and assuming that the norm of such a function is less than the asymptotic gain function of the system for x sufficiently large, it is possible to prove fixed-time convergence to a zone and global boundedness of the system solutions for a nonlinear system (plant) with the same closed-loop setting of the proposed control (5.9) and observer (5.5).

5.2 Parameter Tuning

In this section we provide effective algorithms to tune the parameters involved in the feedback controller. Based on the ILF approach, these algorithms transform the design procedure into an LMI feasibility problem, which simplifies significantly its practical applicability. In addition, they will provide an adjustable upper bound, even if conservative, of the settling-time.

Consider the implicit Lyapunov function candidate

$$Q(V,x) := x^T D_{r(\nu)}(V^{-1}) P D_{r(\nu)}(V^{-1})x - 1, \tag{5.12}$$

where $V \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$ and $P = P^T > 0$. The function Q defined above is very similar to the candidate function introduced in Section 4.1, note that the only difference is that the value of the vector r depends on ν. Besides the properties already discussed about Q, for the controller parametrization we will make use of the property that $Q(1,x) = 0$ implies $x^T Px = 1$ and that with r defined by (5.4), the implicitly defined function $V(x)$ is r-homogeneous of degree 1.

Controller Parametrization

For brevity in the notation, the following representation of system (5.1), (5.2) will be used:

$$\dot{x} = A_0 x + ba\lceil x\rfloor^{\alpha(\nu(\|x\|_P))}, \tag{5.13}$$

where $\lceil x\rfloor^{\alpha(\nu)} = (|x_1|^{\alpha_1(\nu)}\mathrm{sign}(x_1), \ldots, |x_n|^{\alpha_n(\nu)}\mathrm{sign}(x_n))^T$. Note that without loss of generality, the usual norm of the argument of ν in control (5.2) has been replaced with a weighted one (see Remark 5.1). Let us introduce in the notation the matrix $H_{r(\nu)} = -\mathrm{diag}(r_1(\nu), r_2(\nu), ..., r_n(\nu))$. When ν takes a fixed value ν_j denote, for brevity, the homogeneous weights $r_j = r(\nu_j)$, the matrix $H_{r_j} = -\mathrm{diag}(r_{j,1}, r_{j,2}, ..., r_{j,n})$, the dilation matrix $D_{r_j}(\lambda) = \mathrm{diag}(\lambda^{r_{j,1}}, \lambda^{r_{j,2}}, ..., \lambda^{r_{j,n}})$ and the exponent $\alpha_j = \alpha(\nu_j)$. Accordingly,

$$Q_j(V,z) = z^T D_{r_j}(V^{-1}) P D_{r_j}(V^{-1})z - 1. \tag{5.14}$$

Theorem 5.2. *Let for some $\nu_1 \in (-\frac{1}{n}, 0)$, $\nu_2 = -\nu_1$, $\phi, \beta, \kappa, \gamma_j > 0$, $\beta < \phi$ and $\epsilon \in (0,1)$ the system of matrix inequalities*

$$A_0 X + XA_0^T + bY + Y^T b^T + \phi X + \beta bb^T \le 0 \tag{5.15a}$$

$$-\gamma_j X \le H_{r_j} X + XH_{r_j} < 0, \tag{5.15b}$$

$$\xi I_n \le X \le \frac{1}{\kappa} I_n, \quad \begin{pmatrix} \frac{\beta^2 \xi}{\|\bar{z}_j\|^2} & Y \\ Y^T & X \end{pmatrix} \ge 0, \tag{5.15c}$$

$$\bar{z}_{j,i} = \begin{cases} g_{j,i}(\kappa^{-1/2}) + \kappa^{-1/2}\bar{p}_i\epsilon & \text{if } \kappa^{-1/2} \le \alpha_{j,i}^{1/(1-\alpha_{j,i})} \\ \max\{g_{j,i}(\alpha_{j,i}^{1/(1-\alpha_{j,i})}), g_{j,i}(\kappa^{-1/2})\} + \kappa^{-1/2}\bar{p}_i\epsilon, & \text{if } \kappa^{-1/2} > \alpha_{j,i}^{1/(1-\alpha_{j,i})}, \end{cases} \tag{5.16}$$

where $g_{j,i} : \mathbb{R} \to \mathbb{R}$, $g_{j,i}(z) = \left|\lceil z \rfloor^{\alpha_{j,i}} - z\right|$, $\bar{p}_i = (n-i+1)\nu_2$ *and* $\alpha_{j,i} = (\frac{1+n\nu_j}{r_{j,1}}, \ldots, \frac{1+n\nu_j}{r_{j,n}})$, *be feasible for some* $\xi > 0$, $X \in \mathbb{R}^{n\times n}$, $X = X^T > 0$, $Y \in \mathbb{R}^{1\times n}$, *with* $j = 1,2$, $i = \overline{1,n}$.

Then the system (5.1), (5.2) *for* $d = 0$, $a = YP$ *and* $P = X^{-1}$ *is globally fixed-time stable at the origin with*

$$m=\sqrt{\lambda_{\max}(P^{-\frac{1}{2}}D_{r_1}(1-\epsilon)PD_{r_1}(1-\epsilon)P^{-\frac{1}{2}})}<1<M=\sqrt{\lambda_{\min}(P^{-\frac{1}{2}}D_{r_2}(1+\epsilon)PD_{r_2}(1+\epsilon)P^{-\frac{1}{2}})}$$

and the settling-time estimate $T_{\max} \leq \frac{\gamma_1}{(\phi-\beta)|\nu_1|} + \frac{\gamma_2}{(\phi-\beta)\nu_2}$.

The proof of this theorem can be found in the chapter's Proof section. Let us now introduce a constructive procedure, based on Theorem 5.2, to calculate the controller's parameters. This procedure relies in the solution of the system of LMIs (5.15), which can be solved using standard optimization tools such as MATLAB.

Constructive procedure to obtain the controller's parameters P, a, m and M.

1. Set the size of the chain of integrators n.
2. Fix a negative value for ν_1. Start with values close to zero *e.g.* -0.001.
3. Fix positive values for ϵ and κ (a possible initial value is 0.5 for both).
4. Calculate the vectors $\bar{z}_{i,j}$ using (5.16).
5. Fix positive values for ϕ, β, γ_1, γ_2, $\beta < \phi$ (possible starting values are given in the examples).
6. Verify the feasibility of the system of inequalities (5.15).
7. If unfeasible, reduce the value of ϵ or modify the value of κ and repeat from step 3. If feasible, the value of $|\nu_1|$ might be increased in step 2 until a desired value of $T_{\max}$ is obtained without loosing feasibility (recall that in practice, this value might be conservative).
8. From the obtained matrices X and Y, calculate P, a, m and M as described in Theorem 5.2.

Note that the parameters that influence directly the settling-time are $\nu_1, \nu_2, \gamma_1, \gamma_2, \phi$ and β. The parameters ϵ and κ modify the bounds of the inequality (5.15c), so that its manipulation may relax the feasibility conditions of (5.15).

Finally let us remark that following a similar procedure, analogous algorithms can be developed for the observer.

Example 5.2

We start by choosing $n = 3$, $\nu_1 = -0.02$, $\nu_2 = -\nu_1$, $\epsilon = \kappa = 0.5$ and calculating $\bar{z}_j$; following Theorem 5.2 we obtain $\bar{z}_1 = (0.0715, 0.0662, 0.0605)$, $\bar{z}_2 = (0.0721, 0.0689, 0.0661)$. We now choose the parameters $\phi = 0.5$, $\beta = 0.3$, $\gamma_1 = \gamma_2 = 3$ and solve the set of LMIs (5.15a-5.15c) to obtain

$$P_c = \begin{pmatrix} 2.3367 & 1.8430 & 1.7795 \\ 1.8430 & 4.8778 & 2.1295 \\ 1.7795 & 2.1295 & 4.2590 \end{pmatrix}, \quad a = (-0.5952, -1.7576, -1.3889).$$

With this parameter choice, the maximum settling time of the controller is $T_{\max} = 1500s$. Figure 5.3 (left) depicts the substitution of these values in the system (5.1), (5.5) with initial

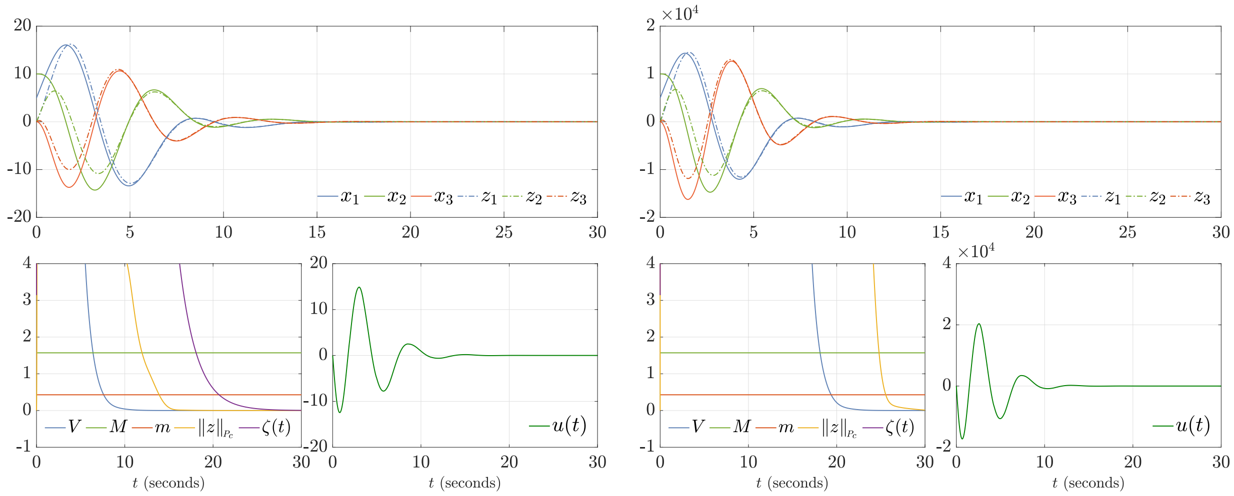

Figure 5.3 – Simulation plot of the unperturbed system (5.1), (5.5) and (5.9) for $n = 3$, using the parameter tuning procedure, initial conditions $x_0 = (0.5, 1, 0)$ (left) and $x_0 = (0.5, 1, 0)$ (right).

conditions $x(0) = (5, 10, 0)$. It is possible to see a fast convergence to the real states before converging also rapidly to the origin. On the right-hand side of Figure 5.3, the initial conditions have been changed to $x(0) = 10^3(5, 10, 0)$, it is possible to see that the settling time is not significantly modified and that in both cases the system reaches the equilibrium long before the settling time estimate. Finally, in Figure 5.4 the initial conditions were reset to $x(0) = (5, 10, 0)$ and the disturbance term

$$d(t) = \sin(2t) + \begin{cases} 10 & \text{if } t \in [30, 31] \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

was added. We can see that the convergence to zero is preserved and with a much better performance than in Example 5.1.

Example 5.3

This last example is meant to compare the fixed-time controller (5.1), (5.2), referred here as the *non-recursive* controller, with the finite-time and fixed-time ones described in [Harmouche et al., 2017], referred accordingly as the *recursive* ones. The parameter choice for the tuning algorithm 5.2 is as follows: $\nu_1 = -1/400, \nu_2 = -\nu_1, \epsilon = 0.1, \phi = 8, \beta = 7, \kappa = 0.5, \gamma_1 = \gamma_2 = 3$. And the obtained parameter values are $a = -(384, 136.08, 16.94)$ and $P_c = 10^3 \begin{pmatrix} 2.3195 & 0.5083 & 0.0359 \\ 0.5083 & 0.1455 & 0.0113 \\ 0.0359 & 0.0113 & 0.0014 \end{pmatrix}$, $\bar{z}_1 = (0.0047, 0.0056, 0.0064)$ and $\bar{z}_2 = (0.0047, 0.0056, 0.0065)$. The corresponding parameters for both the finite-time and the fixed-time controllers in [Harmouche et al., 2017] are the same as the ones used in the example section of the cited article. Figure 5.5 (left) shows the unperturbed states of the chain of integrators with initial conditions $x_0 = (1, 1, 1)$. Clearly, the fixed-time controller outperforms the finite-time one. In Figure 5.5 (right) it is possible to see how in the finite-time case, the variation of initial

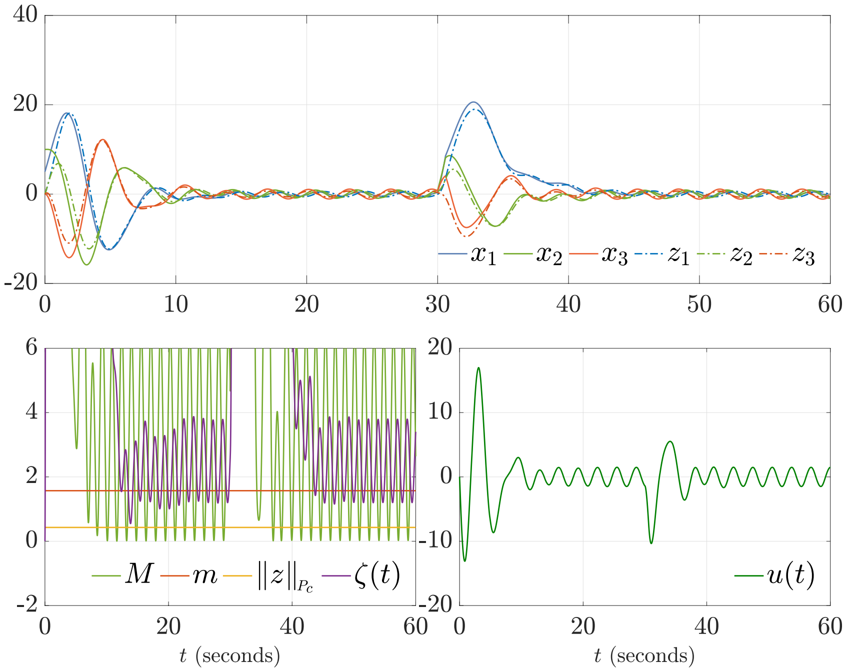

Figure 5.4 – Simulation plot of the perturbed system (5.1), (5.5) and (5.9) for $n = 3$ and using the parameter tuning procedure. The initial conditions are $x_0 = (5, 10, 0)$ and the disturbance term defined by (5.17).

conditions affects the settling-time while the fixed-one remains within two seconds. Finally, Figure 5.6 shows an improvement of over two seconds in the settling time with respect to the recursive fixed-time controller, illustrating that with the tuning algorithm, the settling time can be adjusted. As shown in the interior plots, fixed-time stability is assumed whenever all the states enter a strip of magnitude 10^{-4} (depicted between black lines in Fig. 5.6) and remain there for the rest of the simulation.

5.3 Conclusions

A state feedback control has been constructed for a chain of integrators, which ensures global convergence of all trajectories to the origin with an upper bound of the settling-time that is independent of the initial conditions *i.e.* fixed-time stability. An observer has been proposed, which provides a global estimation of the plant state (global differentiation) with a fixed-time convergence rate. Both control and estimation algorithms are robust with respect to disturbances and noises. It has been shown that the combination of these algorithms results in a global fixed-time output stabilization control law. Effective tools to optimize the scheme's parameters, allowing the maximum settling time to be estimated have been presented and the efficacy of this scheme has been demonstrated in simulations. In the perturbed case, practical fixed-time stabilization is obtained, this is fixed-time stabilization to a ball containing the origin whose radius depends on the size of the perturbation.

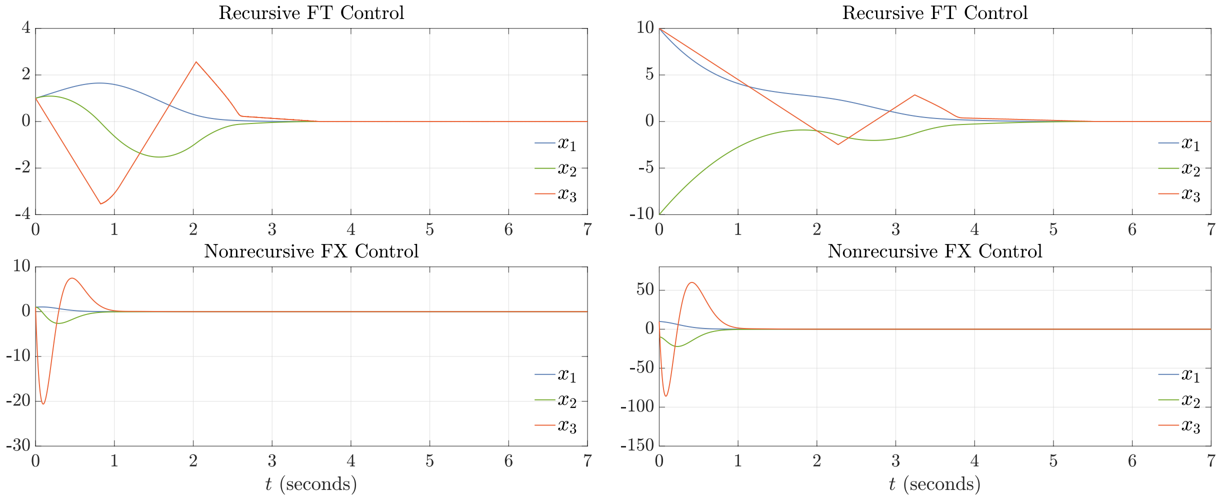

Figure 5.5 – Comparative plots between the recursive finite-time observer and the nonrecursive fixed-time one for $n = 3$, without disturbances and with initial conditions $x_0 = (1, 1, 1)$ (left) and $x_0 = (10, -10, 10)$ (right).

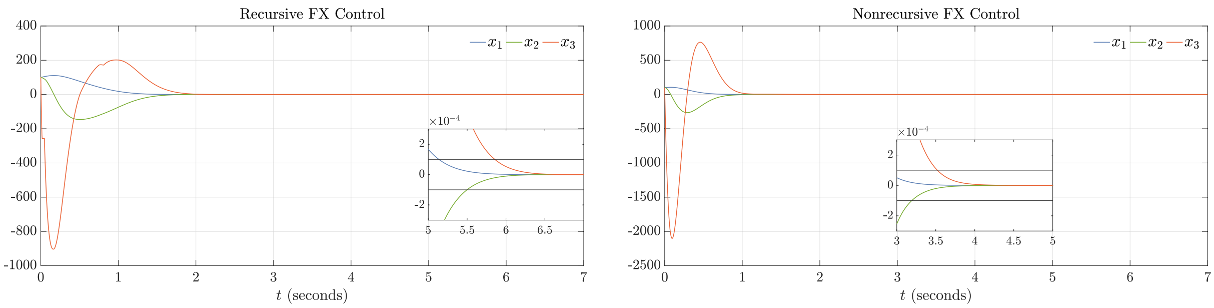

Figure 5.6 – Comparison results between the recursive finite-time observer and the nonrecursive fixe-time one for $n = 3$, without disturbances and with initial conditions $x_0 = (10, 10, 10)$.

As with many control algorithms that focus on *fast* convergence rates, the control signal magnitude might grow significantly to cope with the time constraints imposed. In practice, these constraints can only be granted locally due to boundedness of the admissible control magnitude. However, in contrast to other control algorithms, the FxT controllers do not need to be re-tuned if the admissible control magnitude (and, consequently, the domain of fixed-time convergence) is increased. A related issue deals with the methods used for simulation, which have to be adapted to treat highly nonlinear systems [Efimov et al., 2017].

5.4 Proofs

Proof of Theorem 5.2. **I** The functions $Q_j(V,x)$, $j=1,2$ defined in (5.14) satisfy the conditions **C1-C3** of Theorem 1.13. Indeed, they are continuously differentiable for all $V\in\mathbb{R}_+$ and for all $x\in\mathbb{R}^n$. Since $P>0$, the inequalities

$$\frac{\lambda_{\min}(P)\|x\|^2}{\max\{V^{2\min r_i},V^{2\max r_i}\}}\le Q_j(V,x)+1\le\frac{\lambda_{\max}(P)\|x\|^2}{\min\{V^{2\min r_i},V^{2\max r_i}\}},$$

imply that for any $x\in\mathbb{R}^n\backslash\{0\}$ there exist $V^-\in\mathbb{R}_+$ and $V^+\in\mathbb{R}_+$ such that $Q_j(V^-,x)<0<Q_j(V^+,x)$. Moreover, if $Q_j(V,x)=0$ then, obviously, the condition **C3** of Theorem 1.13 holds and there exists a Lyapunov function candidate $V:\mathbb{R}^n\to\mathbb{R}$ implicitly defined by the identity $x^TD_{r_j}(V^{-1})PD_{r_j}(V^{-1})x=1$.

II Since

$$\partial_VQ_j=V^{-1}x^TD_{r_j}(V^{-1})(PH_{r_j}+H_{r_j}P)D_{r_j}(V^{-1})x,$$

taking into account (5.15b), we have that $\partial_VQ_j<0\ \forall V\in\mathbb{R}_+$ and $x\in\mathbb{R}^n\backslash\{0\}$. So the condition **C4** of Theorem (1.13) also holds and therefore $Q_j(V,z)=0$ implicitly defines a proper positive definite Lyapunov function candidate $V:\mathbb{R}^n\to\mathbb{R}$.

III By denoting ∂_xQ_jf as the partial derivative of Q_j with respect to x along the trajectories of (5.13) we obtain

$$\partial_xQ_jf=2x^TD_{r_j}(V^{-1})PD_{r_j}(V^{-1})[A_0x+ba\lceil x\rfloor^{\alpha(\nu)}].$$

By adding and subtracting inside the brackets the auxiliary term $V^{1+n\nu_j}baD_{r_j}(V^{-1})x$ and taking into account that $D_{r_j}(V^{-1})A_0D_{r_j}^{-1}(V^{-1})=V^{-\nu_j}A_0$ and that $D_{r_j}(V^{-1})b=V^{-1-(n-1)\nu_j}b$, we simplify the derivative as

$$\partial_xQ_jf=\begin{pmatrix}D_{r_j}(V^{-1})x\\ d(V,x)\end{pmatrix}^T\begin{pmatrix}V^{-\nu_j}(PA_0+A_0^TP+Pba+a^Tb^TP) & Pb\\ b^TP & 0\end{pmatrix}\begin{pmatrix}D_{r_j}(V^{-1})x\\ d(V,x)\end{pmatrix},$$

where $d(V,x)=V^{-\nu_j}\left(a(V^{-1-n\nu_j}\lceil x\rfloor^{\alpha(\nu)}-D_{r_j}(V^{-1})x)\right)$. By adding and subtracting the terms $\phi V^{-\nu_j}x^TD_{r_j}(V^{-1})PD_{r_j}(V^{-1})x$ and $\frac{1}{\beta}V^{\nu_j}d^T(V,x)d(V,x)$ we obtain

$$\partial_x Q_j f = \begin{pmatrix} D_{r_j}(V^{-1})x \\ d(V,x) \end{pmatrix}^T \Theta \begin{pmatrix} D_{r_j}(V^{-1})x \\ d(V,x) \end{pmatrix} - \phi V^{-\nu_j} x^T D_{r_j}(V^{-1}) P D_{r_j}(V^{-1})x + \tfrac{V^{\nu_j}}{\beta} d^T(V,x) d(V,x),$$

where

$$\Theta = \begin{pmatrix} V^{-\nu_j}(PA_0 + A_0^T P + Pba + a^T b^T P + \phi P) & Pb \\ b^T P & \frac{V^{\nu_j}}{\beta} \end{pmatrix}.$$

Using the Schur complement, $\left(\begin{smallmatrix} P^{-1} & 0 \\ 0 & \eta \end{smallmatrix}\right)^T \Theta \left(\begin{smallmatrix} P^{-1} & 0 \\ 0 & \eta \end{smallmatrix}\right)$, for any $\eta \in \mathbb{R}$, it is equivalent to the left-hand side of (5.15a) and we arrive to

$$\partial_x Q_j f \leq -V^{-\nu_j} \phi x^T D_{r_j}(V^{-1}) P D_{r_j}(V^{-1})x + V^{\nu_j} \beta^{-1} d^T(V,x) d(V,x).$$

The term $d(V,x)$ can be rewritten as $d(V,x) = V^{-\nu_j} d_0(V,x)$ where $d_0(V,x) = a(V^{-1-n\nu_j} \lceil x \rfloor^{\alpha(\nu)} - D_{r_j}(V^{-1})x)$, and we have that

$$\partial_x Q_j f \leq -V^{-\nu_j} \phi x^T D_{r_j}(V^{-1}) P D_{r_j}(V^{-1})x + V^{-\nu_j} \beta^{-1} d_0^T(V,x) d_0(V,x).$$

If the latter term is bounded by $d_0^T(V(x),x) d_0(V(x),x) < \beta^2$ then we derive

$$\partial_x Q_j f \leq -\phi V^{-\nu_j} + \beta V^{-\nu_j},$$

and using inequality (5.15b) we arrive to

$$\partial_x Q_j f \leq \tfrac{\phi - \beta}{\gamma_j} V^{1+\nu_j} \partial_V Q_j,$$

for $(V,x) : x^T D_{r_j}(V^{-1}) P D_{r_j}(V^{-1})x = 1$. Then, by defining $c_j = \frac{\phi-\beta}{\gamma_j}$, the conditions of Theorem 1.15 are satisfied.

IV Proof of the estimate $d_0^T(V,x) d_0(V,x) \leq \beta^2$.

IV.a Let us introduce the sets $\Omega_1 = \{x : m^2 \leq x^T P x \leq 1\}$ and $\Omega_2 = \{x : 1 \leq x^T P x \leq M^2\}$. Considering first the set Ω_2 and using the change of variables $y = D_{r_2}(V^{-1})x$ we obtain

$$\max_{\substack{(V,x): x^T D_{r_2}(V^{-1}) P D_{r_2}(V^{-1})x = 1 \\ \text{and } x^T P x \in [1, M^2]}} V = \max_{\substack{(V,y): y^T P y = 1 \\ \text{and } y^T D_{r_2}(V) P D_{r_2}(V) y \in [1, M^2]}} V = 1 + \epsilon.$$

Indeed, from inequality (5.15b) we know that $\frac{\partial}{\partial V} y^T D_{r_2}(V) P D_{r_2}(V) y > 0$ and therefore

$$\min_{y: y^T P y = 1} y^T D_{r_2}(V) P D_{r_2}(V) y > \min_{y: y^T P y = 1} y^T D_{r_2}(1+\epsilon) P D_{r_2}(1+\epsilon) y = M^2$$

if $V > 1+\epsilon$ and $1 < \|x\|_p \leq M$ implies $1 \leq V(x) \leq 1+\epsilon$. Similarly, we show

$$\min_{\substack{(V,x):x^T D_{r_1}(V^{-1})PD_{r_1}(V^{-1})x=1 \\ \text{and } x^T Px\in[m^2,1]}} V = 1-\epsilon.$$

Hence, we immediately conclude that $m \leq \|x\|_p^2 \leq M \Rightarrow 1-\epsilon \leq V(x) \leq 1+\epsilon$.

IV.b Recall that the function ν is defined as follows

$$\nu(\|x\|_p) = \begin{cases} \nu_1, & \text{if } \|x\|_p \leq m, \\ \nu_2, & \text{if } \|x\|_p \geq M \\ \frac{\nu_2-\nu_1}{M-m}\|x\|_p + \frac{M\nu_1 - m\nu_2}{M-m}, & \text{otherwise} \end{cases}.$$

Then we have that

$$\begin{aligned} \max_{x\in\Omega_2} d_0^T(V(x),x)d_0(V(x),x) &= \max_{x\in\Omega_2} \delta_\epsilon(V(x),x)^T a^T a\delta_\epsilon(V(x),x) \\ &\leq aa^T \max_{x\in\Omega_2} \delta_\epsilon(V(x),x)^T \delta_\epsilon(V(x),x), \end{aligned}$$

where

$$\delta_\epsilon(V,x) = V^{-1-n\nu_2}\Big(\lceil x \rfloor^{\alpha(\nu(\|x\|_p))} - D_{r_2}(V^{-1})x\Big).$$

Using again the change of variables $y = D_{r_2}(V^{-1})x$ and considering that $x \in \Omega_2 \implies V \in [1,1+\epsilon]$ we have that

$$aa^T \max_{x\in\Omega_2} \delta_\epsilon(V(x),x)^T \delta_\epsilon(V(x),x) \leq aa^T \max_{\substack{V\in[1,1+\epsilon] \\ \|y\|_p=1}} \Xi(V,y)^T \Xi(V,y),$$

where $\Xi(V,y) = \Big|V^{-1-n\nu_2}\lceil D_{r_2}(V)y\rfloor^{\alpha(\nu)} - y\Big|$, and in a component-wise expression we have

$$\Xi_i(V,y) = \Big|V^{-1-n\nu_2+r_{2,i}\alpha_i(\nu)}\lceil y_i\rfloor^{\alpha_i(\nu)} - y_i\Big| = V^{-p_i(\nu)}\Big|\lceil y_i\rfloor^{\alpha_i(\nu)} - V^{p_i(\nu)}y_i\Big|,$$

where $p_i(\nu) = 1 + n\nu_2 - r_{2,i}\alpha_i(\nu) \geq 0$. Hence,

$$\begin{aligned} \Xi_i(V,y) &\leq \max_{\substack{V\in[1,1+\epsilon] \\ \|y\|_p=1}} \Big|\lceil y_i\rfloor^{\alpha_i(\nu)} - V^{p_i(\nu)}y_i\Big| \\ &= \max_{y:y^T Px=1} \Big\{\Big||y_i|^{\alpha_i(\nu)} - |y_i|\Big|, \Big||y_i|^{\alpha_i(\nu)} - (1+\epsilon)^{p_i(\nu)}|y_i|\Big|\Big\} \\ &= \max_{y:y^T Px=1} \Big\{\Big|\lceil y_i\rfloor^{\alpha_i(\nu)} - y_i\Big|, \Big||y_i|^{\alpha_i(\nu)} - y_i + y_i - (1+\epsilon)^{p_i(\nu)}y_i\Big|\Big\} \\ &\leq \max_{y:y^T Px=1} \Big||y_i|^{\alpha_i(\nu)} - |y_i|\Big| + |y_i|\Big|1-(1+\epsilon)^{p_i(\nu)}\Big|. \end{aligned}$$

Lemma 5.3. *The function* $g : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ *defined as* $g(s,\varepsilon) = \Big|s^\varepsilon - s\Big|$, $s,\varepsilon \in \mathbb{R}_{\geq 0}$ *admits the following*

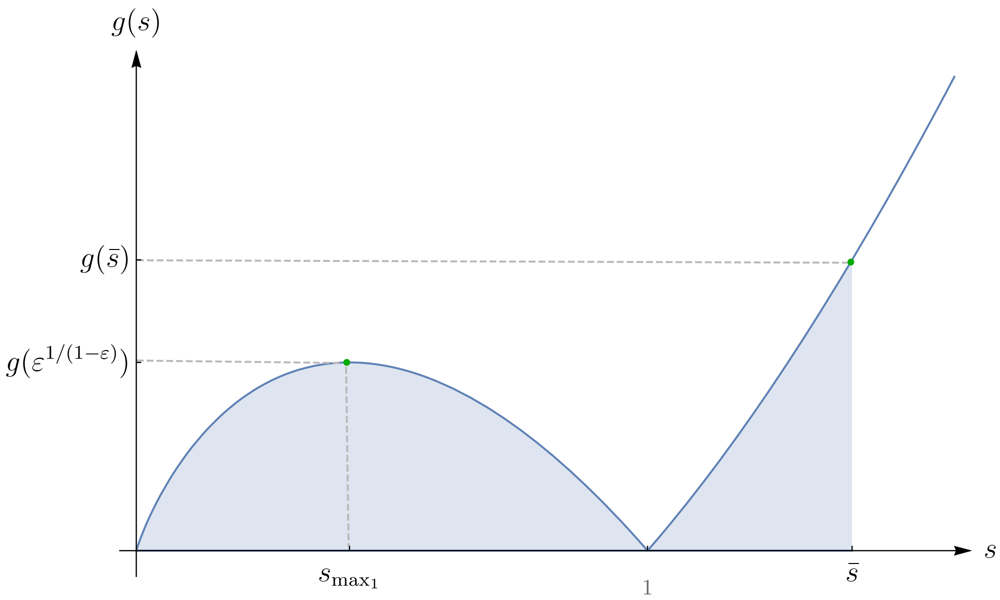

Figure 5.7 – Plot of function $g(s,\varepsilon) = \left|\lceil s \rfloor^{\varepsilon} - s\right|$, with fixed $\varepsilon \in \mathbb{R}_+$. Remark that if $\bar{s} > 1$, the maximum value of $g(s)$ in the interval $s \leq \bar{s}$ is the maximum between $g(\varepsilon^{1/(1-\varepsilon)}, \varepsilon)$ and $g(\bar{s}, \varepsilon)$.

estimate

$$\max_{s\in[0,\bar{s}],\varepsilon\in[0,\bar{\varepsilon}]} g(s,\varepsilon) = \begin{cases} g(\bar{s},\bar{\varepsilon}), & \textit{for } \bar{s} \leq \bar{\varepsilon}^{1/(1-\bar{\varepsilon})} \\ \max\{g(\bar{\varepsilon}^{1/(1-\bar{\varepsilon})},\bar{\varepsilon}), g(\bar{s},\bar{\varepsilon})\}, & \textit{for } \bar{s} > \bar{\varepsilon}^{1/(1-\bar{\varepsilon})}. \end{cases} \tag{5.18}$$

Proof. The function g is depicted in Figure 5.7. It is easy to show that the function g_ε attains a local maximum at $s_{\max_1} = \varepsilon^{1/(1-\varepsilon)}$ within the interval $s \in [0,1]$, therefore, if $\bar{s} \leq s_{\max_1}$ then $\max_{s\in[0,\bar{s}]} g(s,\varepsilon) = g(\bar{s},\varepsilon)$ and if $\bar{s} > s_{\max_1}$ then $\max_{s\in[0,\bar{s}]} g(s,\varepsilon) = \max\{g(\varepsilon^{1/(1-\varepsilon)},\varepsilon), g(\bar{s},\varepsilon)\}$. Taking into account that $\frac{\partial(s^\varepsilon - s)}{\partial\varepsilon} = s^\varepsilon \ln s$, we complete the proof. ∎

Since $y^T P y = 1$ implies that $|y_i| \leq \lambda_{\min}^{-1/2}(P) = \kappa^{-1/2}$, then using Lemma 5.3 the term $\left||y_i|^{\alpha_i(\nu)} - |y_i|\right|$ can be bounded as

$$\left|\lceil y_i \rfloor^{\alpha_i(\nu)} - y_i\right| \leq \tilde{z}_{2,i} = \begin{cases} g_{\alpha_{2,i}}(\kappa^{-1/2}), & \text{for } \kappa^{-1/2} \leq \alpha_{2,i}^{1/(1-\alpha_{2,i})} \\ \max\{g_{2,i}(\alpha_{2,i}^{1/(1-\alpha_{2,i})}), g_{2,i}(\kappa^{-1/2})\}, & \text{for } \kappa^{-1/2} > \alpha_{2,i}^{1/(1-\alpha_{2,i})}. \end{cases}$$

Then for $V \in [1, 1+\epsilon]$ and $y^T P y = 1$ one has

$$\Xi_i(V,y) \leq \tilde{z}_{2,i} + \kappa^{-1/2}((1+\epsilon)^{p_i(\nu)} - 1) \leq \bar{z}_{2,i} + \kappa^{-1/2}((1+\epsilon)^{\bar{p}_i} - 1),$$

where $\nu = \nu(\|x\|_p) \in [0, \nu_2]$, $p_i(\nu) = 1 + n\nu_2 - (1+(i-1)\nu_2)\frac{1+n\nu}{1+(i-1)\nu}$ and $\bar{p}_i = p_i(0) = (n-i+1)\nu_2$. Here, the fact that for any x in the interval $1 \leq \|x\|_p \leq M$, $p_i(\nu(\|x\|_p)) \leq \bar{p}_i$ has been used.

By applying the mean value theorem with $h : \mathbb{R} \to \mathbb{R}$, $h(\theta) = \theta^{\bar{p}_i}$, we obtain that $h(1+\epsilon) - h(1) = h'(\theta^*)\epsilon$, where $\theta^* \in [1, 1+\epsilon]$. Noting that $0 < \bar{p}_i < 1$, we have that $(1+\epsilon)^{\bar{p}_i} - 1 \leq \bar{p}_i\epsilon$ and we arrive to the following estimate

$$\Xi_i(V,y) \le \tilde{z}_{2,i} + \kappa^{-1/2}\bar{p}_i\epsilon,$$

for $V \in [1, 1+\epsilon]$ and $\|y\|_p = 1$. Therefore, it has been proven that

$$\max_{x\in\Omega_2} d_0^T(V(x),x)d_0(V(x),x) \le a^T a\|\bar{z}_2\|^2.$$

where $\bar{z}_2 = (\bar{z}_{2,1}, ..., \bar{z}_{2,n})^T \in \mathbb{R}^n_+$ and $\bar{z}_{2,i} = \tilde{z}_{2,i} + \kappa^{-1/2}\bar{p}_i\epsilon$.

Accordingly, proceeding in the same fashion for the set Ω_1, we obtain

$$\max_{x\in\Omega_1\cup\Omega_2} d_0^T(V(x),x)d_0(V(x),x) \le a^T a\|\bar{z}_2\|^2.$$

Using the Schur complement, inequality (5.15c) becomes $aP^{-1}a^T \le \frac{\xi\beta^2}{\|\bar{z}_j\|^2}$ and since $aa^T\xi \le aP^{-1}a^T$, we have that $aa^T \le \frac{\beta^2}{\|\bar{z}_j\|^2}$ and we conclude that

$$d_0(V(x),x)^T d_0(V(x),x) \le \beta^2$$

if $x \in \Omega_1 \cup \Omega_2$.

IV.c Boundedness of $d_0^T(V(x),x)d_0(V(x),x)$ in $\mathbb{R}^n\backslash(\Omega_1\cup\Omega_2)$

If $x \in \mathbb{R}^n\backslash\{x : \|x\|_p \le M\}$ then $\nu(\|x\|_p) = \nu_2$ and

$$d_0(V(x),x) = aV^{-1-n\nu_2}(\lceil x \rfloor^{\alpha(\nu_2)} - D_{r_2}(V^{-1})x) = a(\lceil y \rfloor^{\alpha(\nu_2)} - y)$$

where $y = D_{r_2}(V^{-1}(x))x$ and $y^T P y = 1$ (since $Q(V(x),x) = 0$). Hence, $\Xi_i(V,y) \le \tilde{z}_{2,i}$ and the required estimate $d_0(V,x)^T d_0(V,x) \le \beta^2$ is straightforward. Similar considerations can be provided for $x : \|x\|_p \le m$. ∎

Conclusions

The work here presented can be mainly seen as contributions to the study of NonA stability of dynamical systems. The results obtained can be divided into two groups: analysis and design results.

The second chapter dealt, using Lyapunov analysis, with the mathematical conditions satisfied by dynamical systems with fixed-time stable origins. Both cases, general and continuous settling-time function, were addressed and a complete characterization (necessary and sufficient conditions) of the FxTS property were obtained by means of a pair of functions. More constructive conditions for FxTS with continuous settling-time functions were also presented and in order to obtain a converse result, the concept of uniform FxT was introduced. These results were further used to find a sufficient condition for fixed-time stabilization of general nonlinear systems that are affine in the control input.

The third chapter addressed the property of NonA ISS. This property provides qualitative means to study both the robustness and the convergence rate of a given input system. Both the explicit and the implicit Lyapunov framework were developed.

In the fourth chapter, the implicit Lyapunov approach was applied to design NonA observers for MIMO linear systems. By using a canonical decomposition, finite-time and fixed-time observation algorithm were obtained and its design procedure relies on an LMI feasibility problem. The robustness of the observers was proved using known results about ISS of homogeneous systems.

Finally, the last chapter presents an output control that achieves FxTS using a sign switching technique of the homogeneous degree. The ILF approach was used to obtain a parametrization algorithm that allows to adjust the settling-time estimate. As in Chapter 4, robustness with respect to measurement noises and perturbations was verified using ISS properties of homogeneous systems.

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Appendix

Appendix A

Publications

A.1 Conference Articles

ECC16, Aalborg, Denmark.

F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti [2016a]. "Finite-time and fixed-time observers design via implicit Lyapunov function". In: *Control Conference (ECC), 2016 European*. IEEE, pp. 289–294

CDC16, Las Vegas, Nevada, USA.

F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti [2016b]. "Fixed-time output stabilization of a chain of integrators". In: *Decision and Control (CDC), 2016 IEEE 55th Conference on*. IEEE, pp. 3886–3891

ECC2018, Limassol, Cyprus.

F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti [2018d]. "On Necessary and Sufficient Conditions for Fixed-Time Stability of Continuous Autonomous Systems". In: *Proc. 17th European Control Conference (ECC)*

CDC2018, Miami Beach, Florida, USA. (accepted)

F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti [2018c]. "On Implicit Finite-Time and Fixed-Time ISS Lyapunov Functions". In: *Decision and Control (CDC), IEEE 55th Conference on*

A.2 Journal Articles

Automatica, 2018.

F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti [2018a]. "Finite-time and fixed-time observer design: Implicit Lyapunov function approach". In: *Automatica* 87, pp. 52–60

International Journal of Robust and Nonlinear Control, 2018.

F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti [2018b]. "Fixed-Time Output Stabilization and Fixed-Time Estimation of a Chain of Integrators". In: *International Journal of Robust and Nonlinear Control*

Control and Estimation in Finite-Time and in Fixed-Time via Implicit Lyapunov Functions

Abstract

This work presents new results on analysis and synthesis of finite-time and fixed-time stable systems, a type of dynamical systems where exact convergence to an equilibrium point is guaranteed in a finite amount of time. In the case of fixed-time stable system, this is moreover achieved with an upper bound on the settling-time that does not depend on the system's initial condition.

Chapters 2 and 3 focus on theoretical contributions; the former presents necessary and sufficient conditions for fixed-time stability of continuous autonomous systems whereas the latter introduces a framework that gathers ISS Lyapunov functions, finite-time and fixed-time stability analysis and the implicit Lyapunov function approach in order to study and determine the robustness of this type of systems.

Chapters 4 and 5 deal with more practical aspects, more precisely, the synthesis of finite-time and fixed-time controllers and observers. In Chapter 4, finite-time and fixed-time convergent observers are designed for linear MIMO systems using the implicit approach. In Chapter 5, homogeneity properties and the implicit approach are used to design a fixed-time output controller for the chain of integrators. The results obtained were verified by numerical simulations and Chapter 4 includes performance tests on a rotary pendulum.

Keywords: finite-time stability, fixed-time stability, homogeneous systems, analyse de lyapunov

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Contrôle et Estimation en Temps Fixe et en Temps Fini via Fonctions de Lyapunov Implicites

Résumé

Dans ce travail, on montre des nouveaux résultats pour l'analyse et la synthèse des systèmes stables en temps fini et fixe. Ce genre des systèmes convergent exactement à un point d'équilibre dans une quantité du temps qui est fini et, dans le cas de systèmes stables en temps fixe, dans un temps maximal constant qui ne dépend pas des conditions initiales du système.

Les chapitres 2 et 3 portent sur des résultats d'analyse ; ce premier present des conditions nécessaires et suffisants pour la stabilité en temps fixe des systèmes autonomes continues tandis que ce dernier combine l'approche de la fonction implicite de Lyapunov avec des résultats de stabilisation ISS pour étudier la robustesse de ce genre de systèmes.

Les chapitres 4 et 5 présentent des résultats pratiques liés á la procédure de synthèse des contrôleurs et des observateurs. Le chapitre 4 emploie la méthode de la fonction de Lyapunov implicite afin d'obtenir des observateurs convergents en temps fini et fixe pour les systèmes linéaires MIMO. Le chapitre 5 utilise des propriétés d'homogénéité et des fonctions de Lyapunov implicites pour synthétiser un contrôleur de sortie en temps fixe pour une chaîne d'intégrateurs. Les résultats obtenus ont été validés par des simulations numériques et le chapitre 4 contient des tests de performance sur un pendule rotatif.

Mots clés : stabilisation en temps fini, stabilisation en temps fixe, systèmes homogènes, lyapunov analysis